T W O "M O D A L" L O G I C S

A D R I A N R E Z U S

Meyhorst 91-01,
6537 KJ Nijmegen
The Netherlands.

N i j m e g e n
August 28, 1990
[r e v i s e d]

TWO 'MODAL' LOGICS*

"Marco Polo describes a bridge, stone by stone.
But which is the stone that supports the bridge? Kublai Khan asks.
The bridge is not supported by one stone or another, Marco answers, **but by the line of the arch they form.**
Kublai Khan remains silent, reflecting. Then he adds: **Why do you speak of the stones? It is only the arch that matters to me.**
Polo answers: **Without stones there is no arch.**"
[Italo Calvino: ***Le città invisibili*** (1972),
English translation by William Weaver.]

This paper concerns two logics of a unusual kind. As propositional logics, they are called **L!4** and **L!5** here. Conveniently, an appended **Q** will distinguish *first-order* variants. **L!4** and **L!5** may be called "modal" because they look so, at least syntactically (like Lewis' **S4** and **S5**). However, unlike for **S4** and **S5**, the **L!**-boxes and diamonds do *not* make up a *raison d'être*. Such logics are "linear" in a sense that can be made precise by type-theoretic methods and play an important rôle in the *local* analysis of the proof-theory of Heyting's logic **H**(**Q**), on the one side, and of classical logic **C**(**Q**), on the other. As expected, *local* is to be contrasted with *global*. These qualifiers apply to proof-theoretic points of view. In particular, "local" points out to the fact that the "global" *syncategoremata* of classical (etc.) logic **if** [...**then**...], **and**, **or**, **every/all** and so forth, might *not* be *ultimate (operational) atoms of meaning* as disclosed by the usual Gentzen *N*-rules. This doubt is justified mainly by the fact that no satisfactory *semantics of classical (logic) proofs* has been provided so far and perhaps also by the fact that we have only *partial* insights into **HQ**-proof behaviors (the "Heyting semantics").

In these notes I will provide only *introductory* comments on the *provability syntax* of **L!4**(**Q**) and **L!5**(**Q**). This is a *proper part* – the easy one – of a larger enterprise that might be characterized, provisionally, as a pure *operational interpretation* of both "intuitionistic" [Johansson-Heyting] and classical proof-theory. The details are deferred and will appear elsewhere.

§1 Syntax. As first-order logics, **L!4Q**, **L!5Q** share the same *syntax*:

- *atoms*:
  - "indeterminates": $F[\![u_1, \ldots, u_n]\!]$, possibly with parameters u_i ranging over *some* universe/domain of "individuals" $\boldsymbol{U}$ (all this may remain anonymous, in the end),
  - "propositional" constants: $\top$ (*summum verum*), $\bot$ (*summum falsum*),

- *logical constants*:
  - *linear entailment* $\rightarrow$ (**linearly entails**),
  - *bi-linear conjunction* & (**with**),
  - *universal exponentiation* $\Box$ (**of course**),
  - *a linear **all**-quantifier* $\forall$ (**[for] all**)

- *formulas*: defined inductively from *atoms*, by closing under the primitive propositional connectives and $\forall$; the spelling is: $(A \rightarrow B)$, $(A \,\&\, B)$, $(A^{\ulcorner\urcorner})$, $\forall u.A$; parentheses and separating dots are used for emphasis, with usual Church-Turing conventions.

The remaining connectives and $\exists$ are introduced by *definition*:

[Df$^-$]	A^-	:=	$A \rightarrow \bot$	(*linear negation*, "*ortho-inversion*"),
[Df**t**]	**t**	:=	$\bot^-$	(*linear verum*),
[Df**f**]	**f**	:=	$\top^-$	(*linear falsum*)
[Df$\Box$]	$A \Box B$	:=	$A^- \rightarrow B$	(*linear fission*, **par**),
[Df$\boxtimes$]	$A \boxtimes B$	:=	$(A \rightarrow B^-)^-$	(*linear fusion*),
[Df$\oplus$]	$A \oplus B$	:=	$(A^- \,\&\, B^-)^-$	(*bi-linear disjunction*),
[Df$\Diamond$]	$A^{<>}$	:=	$A^{-\ulcorner\urcorner-}$	(*existential exponentiation*, **why not**).
[Df$\exists$]	$\exists u.A$	:=	$(\forall u.A^-)^-$	(*linear existence*).

In notation, the ortho-inversion has strongest binding power; e. g., in [Df$\Diamond$], parentheses are to be restored as in $((A^-)^{\ulcorner\urcorner})^-$.

[Proto-] linear equivalence may be thought of as an abbreviation or as a connective, I also introduce *ortho-symmetry* (double ortho-linear negation) in the same way:

[Df$\leftrightarrow$]	$A \leftrightarrow B$	:=	$(A \rightarrow B) \,\&\, (B \rightarrow A)$,	
[Df$^=$]	$A^=$	:=	A^{--},	[actually $(A^-)^-$].

The two *exponentials* $\Box$ and $\Diamond$ are *super-scripted* and are also said to be *modalities*. Up to a certain point, they will behave like $\Box$ and $\Diamond$ resp. in Lewis' logics **S4** and **S5**. One defines also connectives,

[Df$\Rightarrow$]	$A \Rightarrow B$	:=	$A^{\ulcorner\urcorner} \rightarrow B$,
[Df$\wedge$]	$A \wedge B$	:=	$A \,\&\, B$,
[Df$\vee$]	$A \vee B$	:=	$A^{\ulcorner\urcorner} \oplus B^{\ulcorner\urcorner}$,
[Df$\neg$]	$\neg A$	:=	$A \Rightarrow$ **f**,

corresponding to the usual intuitionistic (in **L!4**) resp. classical (in **L!5**) analogues, together with a notational expedient called *strong negation*:

[Df~] $A^{\sim} := A \Rrightarrow \bot$.

[One could have defined many more - even modal, without quotes - as suggested recently by Nuel Belnap, in correspondence.]

The propositional part. The propositional systems **L!4** and **L!5** are presented *axiomatically* first; relevant fragments are isolated and labelled as shown, e. g, the fragment called **BCI** is given by {(**b**), (**c**), (**i**), (→)}, **oL** (*ortho-linear logic*) is axiomatized by {(**b**), (**c**), (**i**), ($\mathbf{k}_*$), (Ω), (Δ), (→)}, **pL** (*proto-linear logic*) adds the &-axioms with (&) to **oL**, etc. This taxonomy is convenient (and has even a good motivation).

Further, ==> stands for the meta-theoretic conditional.

(**b**)	[Prefixing]	$A \to B \to .(C \to A) \to (C \to B)$	
(**c**)	[Commutation]	$A \to (B \to C) \to .B \to (A \to C)$	
(**i**)	[Identity]	$A \to A$	
(→)	[*Modus ponens*]	A, $A \to B$ ==> B	
			BCI
(Ω)	[*Summum verum*]	$\top$	
($\mathbf{k}_*$)	[$\top$-Simplification]	$\top \to .A \to \top$	
(Δ)	[*Duplex negatio affirmat*]	$A^{=} \to A$	
			oL
($\mathbf{p}_1$)	[Left projection]	$A \,\&\, B \to A$	
($\mathbf{p}_2$)	[Right projection]	$A \,\&\, B \to B$	
($\mathbf{s}_{\&}$)	[Composition]	$(A \to B) \,\&\, (A \to C) \to .A \to B \,\&\, C$	
(&)	[Adjunction]	A, B ==> A & B	
			pL
($\mathbf{K}_{[]}$)	[Normality]	$(A \to B)^{[]} \to .A^{[]} \to B^{[]}$	
($\mathbf{i}_{[]}$)	[*Ab opportere esse*]	$A^{[]} \to A$	
($\mathbf{4}_{[]}$)	[Lewis_4]	$A^{[]} \to A^{[][]}$	
(□)	[Exponentiation]	A ==> $A^{[]}$	
($\mathbf{k}_{[]}$)	[□-Simplification]	$A \to .B \Rrightarrow A$	
($\mathbf{s}_{[]}$)	[□-Self-distribution]	$A \Rrightarrow (B \to C) \to .(A \Rrightarrow B) \to (A \Rrightarrow C)$	
			L!4
($\mathbf{5}_{[]}$)	[Lewis_5]	$A^{\sim\sim} \to A^{[]}$	
			L!5

The first-order part. Finally, **L!4Q** and **L!5Q** arise from the corresponding propositional segments by adding the following set of "linear" assumptions on the ∀-quantifier:

(i∀)	[Ground generalization]	A⟦u⟧ ==> ∀u.A⟦u⟧, if A is an axiom,
(k∀)	[Atomic generalization]	$A \to \forall u.A$,
($\mathbf{s}_{\to}$∀)	[→-Generalization]	$\forall u.(A \to B) \to .\forall u.A \to \forall u.B$,
($\mathbf{s}_{\&}$∀)	[&-Generalization]	$\forall u.A \,\&\, \forall u.B \to \forall u.(A \,\&\, B)$,
(Φ∀)	[Atomic ∀-Commutation]	$\forall u.\forall v.A \to \forall v.\forall u.A$,
(Θ∀)	[∀-Commutation]	$A \to \forall u.B \to \forall u.(A \to B)$,
(∀)	[Instantiation]	∀u.A⟦u⟧ ==> A⟦u:=**t**⟧.

In (∀), **t** is any term with u not free for **t** in A and, further, in (**k**∀), (Θ∀), u must not be free in A. As expected, ...⟦u:=**t**⟧ stands for a substitution operator.

Notes. Incidentally, one would also like to mention extensions **R!4** and **R!5** of **L!4** and **L!5** resp., by a "un-exponentiated" (s), viz.

(s) [Self-distribution] $A \to (B \to C) \to .(A \to B) \to (A \to C)$.

So, among other things, (s_{c3}) becomes redundant in **R!4** and **R!5**. The relevant fragments of **R!4** will be labelled correspondingly: **oR** := **oL** + (s) and **pR** := **pL** + (s). I won't insist too much on **R!5**, however: it turns out to be a clumsy way of re-formulating Lewis' **S5**. As ever, appending **Q** to the name of a propositional logic yields a name for the corresponding first-order extension.

The axiomatics appearing here contain several redundancies: the main objective has been to get a *separated* formulation for the basic system **L!4Q**, in the "layered" sense, rather than to make some economy. [The **L!5(Q)**-formulation is *not* separated.]

Among other things, at the first-order level, the axiom schemes (Φ∀) and (Θ∀) can be dispensed with, although, for proof-theoretic purposes, it it not so wise to do it [since it is hard to get them back].

Several *remarks* are in order. I will supply historical details whenever applicable.

§2 L!4Q is Girard's "linear" logic. The first thing worth saying at the very beginning is perhaps the fact that **L!4Q** is (equivalent to) Girard's [87] "linear" logic[1].

A proof of equivalence would be useless here: it presupposes some familiarity with Girard [87] and tedious explanations I would rather like to avoid[2].

Subsystems of **L!4Q**. Although *full* **L!4Q** is a "logic of Girard", it has a rich heredity. I will tell the story, once more, here, because most aspects of it, known before Girard, are of a mere archivistic nature, if not even pure folklore.

BCI. The pure implicational part of **L!4Q** is the same as that of **L!4** and is axiomatized by **BCI**. The connective so axiomatized is the *pure linear implication* or the *Smiley-Meredith-Jaśkowski implication*; this deserves some proof. **BCI** is, actually, the most important part of **L!4Q**, since it determines the behavior of the underlying notion of entailment *via* an appropriate Deduction Theorem.

This "logic" is implicit in a note of Alonzo Church [51a] and arose first in speculations about the concept of a would-be "minimal" implication. It appears explicitly in Smiley [55] as $\mathbf{L}_4$[3].

Timothy Smiley has investigated the required form of Deduction Theorem for **BCI**, along Church's suggestions, and established (at least implicitly) that the natural deduction variant of the system requires that each hypothesis must be *used* in derivations and, if so, it must be used *exactly once*.

The name comes from Carew A. Meredith, who isolated it independently (in 1956 or earlier) and provided different axiomatizations for it, even in the single-axiom-*cum*-(MP)-style (see Prior [62], Meredith & Prior [63], Rezus [82]). The motivation for the name is in the *proposition-as-types* interpretation of the axiomatics, rediscovered (in the fifties, - again independently, - after Curry, but *before* many others) by Meredith.

In this view, the axioms are supposed to own "primitive proofs" **b**, **c** and **i**, to be understood as *combinators* in an interpreted typed combinatory language, where *application* is the *Meredith condensed detachment operator*. [As an aside, Meredith's theory can be extended such as to cover *first-order classical logic*, as well.] From incidental comments of Meredith - recorded by Arthur N. Prior in print and otherwise - it is obvious that he has also formulated a natural deduction variant of **BCI**, by imposing the expected restrictions (here: "linearity") on the formation of λ-terms in the corresponding typed λ-calculus.

Somewhat later (1960), Stanislas Jaśkowski has provided a decision procedure for the same system (Jaśkowski [63]). Jaśkowski says that he has been stimulated to study **BCI** (in fact, **BCK**, see below) by Helmuth Thiele.

Models for **BCI** have been found by Alasdair Urquhart, around 1972, (Urquhart [72a], this is an *abstract*, and only *a single line* of it concerns the present comment; full details are, however, given in the dissertation Urquhart [73] and can be also extracted from Urquhart [72,72b], putting abelian monoids in place of semi-lattices; cf. also Urquhart's contributions to Anderson & Belnap [8*] [= **Entailment II**], promised as **§47** already in Anderson & Belnap [75], as well as related information appearing in Urquhart [86] and Dunn [86]).

The *monoidal semantics* for **BCI**, developed by Urquhart extends to the *full* **L!4Q** (*sic*), with some assistance from Girard [87]: this semantics has been rediscovered (in August 1986) by Girard and appears as "semantics of phases" in that monograph[4].

Except the "linearity"-condition on the use of hypotheses in (natural deduction-style-) derivations, mentioned earlier, **BCI** is remarkable syntactically by the fact that its "principal" theorems [i. e., formulas that are not proper subsitution instances of **BCI**-theorems] are classical tautologies where each propositional atom ("variable") occurs *exactly twice* (this has been first observed by Smiley, and later by Jaśkowski).

Less obvious is the fact that *Meredith application* - usually a *partial* operator on formulas (see Rezus [82]) - *is totally defined on* **BCI**-*theorems*. This is, in fact, a feature shared with **BCK**, a purely implicative "logic" which extends **BCI** by

[**k**] [Simplification] $A \to .B \to A$

(also investigated by Meredith and Jaśkowski), conjectured by Carew Meredith, again, in the sixties; an affirmative answer for **BCK** with a *correct* proof has been provided only recently by Roger Hindley (February 1987; the **BCI**-analogue follows, although it could have been also recovered from an earlier remark of Mariangiola Dezani and Mario Coppo).

Ortho- and proto-linear logic. **BCI** *plus* an involutive (say "classical") negation, defined inferentially, in terms of $\bot$ - actually **oL**, if we agree to forget about the somewhat boring axioms (Ω), (**k**$_*$) and the constant $\top$ -, has been investigated by Timothy Smiley [58/9], who defended it philosophically, as a logic of *entailment*, free of so-called "paradoxes of relevance".

The fragment called here **oL** and referred to as *ortho-linear logic* is an exact axiomatization of Girard's "multiplicatives" while **pL**, referred to as *proto-linear logic*, axiomatizes also the behavior of his "additives"[5].

One could also notice the fact that (**k**$_*$) and (Ω) might have been replaced - without any damage as regards the intended separation properties - by

(Θ) [*Ad quodlibet verum*] $A \to \top$.

Contraction. Although **BCI**, **oL** as well as **pL** deserve the name "relevant logics", according to criteria advocated in Anderson & Belnap [75], Routley *et al.* [82], etc., they are actually very weak, due to the absence of Contraction, viz. the Hilbert axiom

(**w**) [Contraction] $A \to (A \to B) \to .\ A \to B$

or, equivalently, of full Self-distribution on the major

(**s**) [Self-distribution] $A \to (B \to C) \to .(A \to B) \to (A \to C)$.

This explains somewhat why even Richard Sylvan (better known as R. Routley), who took some care to isolate systematically most weak relevant logics which might have ever been of some interest, has paid little attention to them in Routley *et al.* [84] and other places.

Logics *without* Contraction have been first investigated by F. B. Fitch (1934 and later) and W. Ackermann (1937 and later) in connection with the occurrence of paradoxes in logics based on combinators and λ-calculi ("illative" logics).

The immediate heuristics motivating such a logic can be extracted from even very superficial an inspection of *Curry's paradox*. But the omission of Contraction leads to foundationally uninteresting systems.

Notably, Łukasiewicz' many-valued systems lack Contraction, too, but retain "non-relevant" principles, as, e. g., (**k**). Apparently, Łukasiewicz was brought to consider such systems by reflections on the nature of modalities in Aristotle. See, e. g., Łukasiewicz [30].

By prohibiting *both* Contraction *and* Weakening - (**w**) and (**k**) say - one obtains a reasonable starting point for a couple of *deviant logics* as, e. g., advocated by *connexivists*, on the one hand (McCall [66]), or by various *para-consistentists*, on the other (see Priest & Routley [*83,84], or Priest *et al.* [8*]).

The logics **L!4Q** and **L!5Q** are, *prima facie*, in the same deviant camp. On second thoughts, they will turn out to be rather imbued of *orthodoxy* and will pretend to advise on *how to do* intuitionistic and classical logic, resp. Their apparent deviance is, in fact, pretty old strategy: *reculer pour mieux sauter*.

§3 "Relevant" neighbors. Adding either (**w**) or (**s**) to **BCI** gives the Moh-Church pure *relevant* system $\mathbf{R}_{\to}$, (Moh [50], Church [51]; this axiomatizes exactly the pure implicational part of the Anderson-Belnap *relevant logic* **R** (Anderson & Belnap [75]).

On the same line of thought, **oL** *plus* (**w**) is (theoremwise) equivalent to a formulation of the implication-*cum*-negation fragment of **R**, with added constants, *à la* Robert Meyer; see details, especially about $\mathbf{R}^{t}_{\to}$, in Anderson & Belnap [75] and the literature cited there.

By analogy with **oL**, **oL** *plus* (**w**) will be called **oR** here (*ortho-R*, *ortho-relevant logic*). Of course, while formulating **oR** axiomaticattly, with [*Modus ponens*], one can take (**s**) in place of (**w**), as well.

On the other hand, **pL** *plus* (**w**) - or *plus* (**s**), if one prefers - has been less popular among students of relevant logics, although it is, in a sense, *better* than **R**, since it lacks the proof-theoretically annoying distributivity principles (typical for **R** or Chidgey's **U**):

[**d**] $A \;\&\; (B \oplus C) \to (A \;\&\; B) \oplus (A \;\&\; C)$, or
[**d⁻**] $A \;\&\; (B \oplus C) \to (A \;\&\; B) \oplus C$.

The latter "logic" axiomatizes (*exactly*) **R** *minus* ($\mathbf{d}^{-}$), at least if one works with the Anderson-Belnap most preferred **R**-axiomatics. This fragment of **R** has got some support recently, in computer science studies, and has been examined, monographically, under the label **LR**, in Thistlewaite *et al.* [87] ("**LR**" stands there, we are told, for *lattice*-**R**, where "lattice" is motivated semantically: **LR** admits of Dunn-style semantics - see Dunn [66]- with *non*-distributive lattices).

A good name for it, respecting, moreover, the previous convention about "**σ**" and "**p**", is **pR**, or *proto-relevant logic*. Proof-theoretically, we might want to call it "bi-linear logic" (it would be too long to explain here why).

§4 Linear quantifiers. At this point, it is advisable to have a quick look (and once forever) to the *proper* first-order part of **L!4Q** and **L!5Q**.

The quantifiers of a "linear" logic are, of course, linear. For instance, the Anderson & Belnap [8*] "confinement"-principles:

(**d**∀)	∀u.(A ⊕ B) → A ⊕ ∀u.B,	where u is not free in A,
(**d**∃)	A & ∃u.B → ∃u.(A & B),	where u is not free in A,

are *non*-linear assumptions on proof-behaviors. [Although there is much more to say, the proper comparison is with (**d**) above.] We may also note that the "linear" quantifiers look, in fact, like those of **pRQ** (the first order **pR**, which has been also said to be proof-theoretically - and somewhat mysteriously - "bi-linear").

In passing, my choice for the ∀-postulates above has its reasons in a combinatory analysis of the corresponding "linear" proofs. However, even without going too deep into proof-theoretic details, one can see that half of my list hides an *induction*.

The [Atomic generalization] (**k**∀) looks very much like the [Ground generalization] rule (**i**∀). Its meaning is different: the postulates (**i**∀) and (**k**∀) make up the basis case of an inductive definition of generalization, with inductive clauses accounted for as ($s_{\to}$∀), ($s_{\&}$∀) ([→- and &-Generalization]) and (⊕∀).

One could have had a more redundant view on generating inductively the appropriate generalization rule; the following schemes can be derived, even in **pLQ**:

($\mathbf{b}_{\to}$∀)	A → B → .∀u.A → ∀u.B,	u not free in A, B,
($\mathbf{c}_{\to}$∀)	∀u.(A → B) → .A → ∀u.B,	u not free in A,
($\mathbf{b}_{\&}$∀)	A & ∀u.B → .∀u.(A &.B),	u not free in A,
($\mathbf{c}_{\&}$∀)	∀u.A & B → .∀u.(A & B),	u not free in B,

and, for some other reason, so is

(@$_{*}$∀) ∀u.(∀v.A⟦v⟧ → A⟦u⟧, (avoiding u/v clashes).

This gives, already in **pLQ**, a full *generalization rule*: for all *U*-terms t,

(↑$_{t}$∀) A⟦u:=**t**⟧ ==> ∀u.A⟦u⟧, where u is not free for **t** in A.

Obviously, taking (↑$_{t}$∀) and (∀) primitive should suffice for **pLQ**, **L!4Q**, etc., as introduced above.

This being said, it is hardly necessary to bring the quantifiers' behavior under focus any more. They will be largely ignored in the rest of the paper. There is a good reason for this: their addition is *conservative* over the propositional fragments (we don't get more pure $\rightarrow$-theorems, say, in **L!4Q** than we had already in **L!4** (and analogously for the **L!5**-case).

Note. This is, probably, not immediately obvious at this level of the discussion. Of course, I can see it because the combinatory machinery behind the axiomatics provides the result automatically. This should be the easy way of showing conservativity. Unfortunately, the details must be deferred. A useful exercise consists of trying to get the same result in the hard way, by models (for **L!5Q**, Girard [87] is *not* useful although Urquhart [86] *might* be).

§5 "Modalities": the interplay between L!4, Lewis' S4 and Heyting's logic. The addition of the *exponentials* □ and ◊ to **pL** opens the doors to quite different a paradise: technically, we add **S4**-like axioms to **pL** in order to obtain **L!4**, but this is only *one half* of the game. So, **L!4** is *a kind of "modal" logic*[6], with a proto-linear basis in place of a classical one.

In order to locate conceptually the *other half* of the game one must first note that **pL** lacks *exactly* Weakening[7] *and* Contraction[8] in order *to be* (equivalent to) Classical Logic. Axiomatically, the missing items are the intuitionistic[9] axioms (**k**) and (**s**). Now we re-introduce them in somewhat restricted, "exponentiated" form, here ($\mathbf{k}_{[]}$) and ($\mathbf{s}_{[]}$) resp.

It is easy to see that the **S4**-characteristic part allows also proving, with some assistance from **BCI**, fully "exponentiated" (**k**) and (**s**), namely

($\mathbf{k}_{[][]}$) [□□-Simplification] $A \Rightarrow .B \Rightarrow A$,
($\mathbf{s}_{[][]}$) [□□-Self-distribution] $A \Rightarrow (B \Rightarrow C) \Rightarrow .(A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

In fact, any "exponentiation" of

($\mathbf{s}_{[]o}$) [$□_o$-Self-distribution] $A \rightarrow (B \rightarrow C) \rightarrow .(A \rightarrow B) \rightarrow (A \Rightarrow C)$,
($\mathbf{w}_{[]o}$) [$□_o$-Contraction] $A \rightarrow (A \rightarrow B) \rightarrow .A \Rightarrow B$

is equally available as a **L!4**-theorem (the "exponentiation" consists, practically, of replacing one or more $\rightarrow$'s by $\Rightarrow$). In particular, so is

($\mathbf{w}_{[]}$) [□-Contraction] $A \rightarrow (A \Rightarrow B) \rightarrow .A \Rightarrow B$.

One shows quite easily (by using appropriate matrices, for instance) that **L!4** is also "modally interesting" (that is, the "modal" axioms ($\mathbf{k}_{[]}$) and ($\mathbf{s}_{[]}$) do *not* collapse $\rightarrow$ into $\Rightarrow$, such that $A \rightarrow B$ and $A \Rightarrow B$ remain, in general, linearly non-equivalent in **L!4**).

Thus, in the end, the "modal" game looks worthwhile playing.

It is equally easy to show that (**b**), (**c**) and (**i**) are also **L!4**-provable in every possible "exponentiated" form, whereas, obviously, $\Rightarrow$ satisfies [*Modus ponens*]. So **L!4** contains at least Johansson Minimal Implication in the form of $\Rightarrow$.

At this point it will be useful to notice that, in the above axiomatization of **L!4**, ($\mathbf{s}_{\ulcorner\urcorner}$) and ($\mathbf{w}_{\ulcorner\urcorner}$) are interchangeable. In fact, one can write down the Meredith-*proof* (Meredith-combinator derivation) of ($\mathbf{s}_{\ulcorner\urcorner}$) using only **BCI**, i. e.

$$\mathbf{s}_{\ulcorner\urcorner} = \mathbf{b}(\mathbf{b}(\mathbf{b}\mathbf{w}_{\ulcorner\urcorner})\mathbf{c})(\mathbf{b}\mathbf{b})$$

or with, $\alpha \circ \beta := \mathbf{b}\alpha\beta$, ignoring "type-parametrizations",

$$\mathbf{s}_{\ulcorner\urcorner} = (\mathbf{b}\mathbf{w}_{\ulcorner\urcorner} \circ \mathbf{c}) \circ \mathbf{b}\mathbf{b},$$

while the easiest way to get ($\mathbf{w}_{\ulcorner\urcorner}$) is with ($\mathbf{k}_{\ulcorner\urcorner}$) and (**i**) or (**c**),

$$\mathbf{w}_{\ulcorner\urcorner} = \mathbf{s}_{\ulcorner\urcorner}\mathbf{s}_{\ulcorner\urcorner}(\mathbf{k}_{\ulcorner\urcorner}\mathbf{i}) = \mathbf{s}_{\ulcorner\urcorner}\mathbf{s}_{\ulcorner\urcorner}(\mathbf{c}\mathbf{k}_{\ulcorner\urcorner}).$$

As an aside, ($\mathbf{k}_{\ulcorner\urcorner}$) is superfluous: one can derive ($\mathbf{w}_{\ulcorner\urcorner}$) from ($\mathbf{s}_{\ulcorner\urcorner}$) using only **BCI**.

A little more reflection shows (Girard [87]) that $\Rightarrow$, $\wedge$, $\vee$ and $\neg$ are, indeed, the Heyting ("intuitionistic") connectives, so intuitionistic propositional logic **H** is contained (properly) in **L!4**.

Girard's translation. In fact, we may think of [Df$\Rightarrow$], [Df$\wedge$], [Df$\vee$] and [Df$\neg$] as defining a *translation of* **H** *into* **L!4**, and analogously for the first-order case. This is actually the *Girard translation* (Girard [87]); I shall denote it here by $(\ldots)^{\llcorner}$. Re-using $\Rightarrow_H$, $\wedge_H$, $\vee_H$, and $\neg_H$, this time as official Heyting connectives in **H**, $(\ldots)^{\llcorner}$ can be defined inductively by:

$$\begin{array}{llll}
(A)^{\llcorner} & = A, & & \text{if } A \text{ is an atom of } \mathbf{H}, \\
(A \Rightarrow_H B)^{\llcorner} & = ((A)^{\llcorner})^{\ulcorner\urcorner} \to (B)^{\llcorner} & = & (A)^{\llcorner} \Rightarrow (B)^{\llcorner}, \\
(A \wedge_H B)^{\llcorner} & = (A)^{\llcorner} \,\&\, (B)^{\llcorner} & = & (A)^{\llcorner} \wedge (B)^{\llcorner}, \\
(A \vee_H B)^{\llcorner} & = ((A)^{\llcorner})^{\ulcorner\urcorner} \oplus ((B)^{\llcorner})^{\ulcorner\urcorner} & = & (A)^{\llcorner} \vee (B)^{\llcorner}, \\
(\neg_H A)^{\llcorner} & = ((A)^{\llcorner})^{\ulcorner\urcorner} \to \mathbf{f} & = & (A)^{\llcorner} \Rightarrow \mathbf{f},
\end{array}$$

Note. Certainly, in the case of quantifiers, we must have the same kind of "commuting" behavior for $(\ldots)^{\llcorner}$: $(\forall_H u.A)^{\llcorner} = \forall u.(A)^{\llcorner}$, but recall that I have already decided to ignore them.

Clearly, the clause for $\neg_H$ amounts to $(A \Rightarrow_H \mathbf{f})^{\llcorner}$, so **f** is the "linear" representation, in **L!4**, of Heyting's *absurdum*.

Gödel translations. As Girard notes incidentally, $(\ldots)^{\llcorner}$ is vaguely reminiscent of the *modal translation* of **H** into **S4** (see Gödel [33]). The differences are, however, *radical*. In order to see this, we may ignore again the quantifiers and pay attention exclusively to the corresponding *propositional* logic-fragments.

First, there are *several* distinct ways of making a "provability-preserving Gödel translation" **H** $\longrightarrow$ **S4** (cf. McKinsey & Tarski [48], Troelstra [86] or details following below).

Next, while doing $(\ldots)^L$ one has a significant *technical* departure from any translation of the Gödel-type. Let's look into the rudiments.

Let $(\ldots)^G$ stand for the *original* Gödel [33] translation.

Writing $\supset$, $\wedge$, $\vee$, $\sim$, resp. for the official classical connectives, with $\sim$, defined inferentially in terms of a *classical falsum* constant **F** and $\Box$, $\dashv$ for the specific **S4**-connectives, with, say, $\Box$ primitive, *à la* Gödel-Lemmon and $\dashv$ given, as usual, by

$$A \dashv B := \Box(A \supset B),$$

the *Gödel translation* $(\ldots)^G$ reads, inductively,

$$
\begin{array}{ll}
(A)^G & = A, \text{ if } A \text{ is an atom of } \mathbf{H}, \text{ with, in particular,}\\
(f_H)^G & = \mathbf{F},\\
(A \Rightarrow_H B)^G & = \Box(A)^G \supset \Box(B)^G,\\
(A \wedge_H B)^G & = (A)^G \wedge (B)^G,\\
(A \vee_H B)^G & = \Box(A)^G \vee \Box(B)^G,\\
(\neg_H A)^G & = \Box(A)^G \supset \mathbf{F} \;=\; {\sim}(\Box(A)^G).
\end{array}
$$

Another Gödel translation, $(\ldots)^{MG}$ say, (for *modified Gödel*, cf. also Gödel [33]), can be defined by changing the clauses for $\wedge_H$ and $\neg_H$ above into

$$
\begin{array}{ll}
(A \wedge_H B)^{MG} & = \Box(A)^{MG} \wedge \Box(B)^{MG},\\
(\neg_H A)^{MG} & = \Box(\Box(A)^{MG} \supset \mathbf{F}) = \Box({\sim}(\Box(A)^{MG})) = \Box(A)^{MG} \dashv \mathbf{F}.
\end{array}
$$

Note that $(\ldots)^{MG}$ is very much like the *McKinsey-Tarski translation*, of McKinsey & Tarski [48]: denoting the latter by $(\ldots)^{MT}$ one has,

$$
\begin{array}{ll}
(A)^{MT} & = \Box A, \text{ if } A \text{ is an atom of } \mathbf{H}, \text{ with, in particular,}\\
(f_H)^{MT} & = \Box\mathbf{F},\\
(A \Rightarrow_H B)^{MT} & = \Box(A)^{MT} \supset \Box(B)^{MT},\\
(A \wedge_H B)^{MT} & = \Box(A)^{MT} \wedge \Box(B)^{MT},\\
(A \vee_H B)^{MT} & = \Box(A)^{MT} \vee \Box(B)^{MT},\\
(\neg_H A)^{MT} & = \Box((A)^{MT} \supset \mathbf{F}) = \Box({\sim}(A)^{MT}) = (A)^{MT} \dashv \mathbf{F}.
\end{array}
$$

The translations $(\ldots)^G$ and $(\ldots)^{MT}$ are such that, for any **H**-formula A, one can prove

$$
\begin{array}{l}
\mathbf{S4} \vdash \Box(A)^G \supset (A)^{MT},\\
\mathbf{S4} \vdash (A)^{MT} \supset \Box(A)^G.
\end{array}
$$

So, in *both* cases one has, with McKinsey & Tarski [48], for any **H**-formula A,

$$\text{(G)} \qquad \mathbf{H} \vdash A \iff \mathbf{S4} \vdash (A)^G \iff \mathbf{S4} \vdash (A)^{MT}.$$

In particular, for any **H**-formula of the form $C := A \Rightarrow_H B$, one has

$$\mathbf{H} \vdash C \iff \mathbf{S4} \vdash \Box(A)^G \supset \Box(B)^G \iff \mathbf{S4} \vdash \Box(A)^{MT} \supset \Box(B)^{MT},$$

which gives,

$$\text{(1G)} \quad \mathbf{H} \vdash C \iff \mathbf{S4} \vdash \Box(A)^G \supset (B)^G \iff \mathbf{S4} \vdash \Box(A)^{MT} \supset (B)^{MT},$$

since

$$\mathbf{S4} \vdash \Box A \supset \Box B \iff \mathbf{S4} \vdash \Box A \supset B.$$

Actually, both $(\ldots)^G$ and $(\ldots)^{MT}$ are **S4**-*neutral*: the use of **S4** is non-specific. Indeed, one can replace **S4** in (G), by **S3** (see Hacking [63]) or by **S4Grz**, where **S4Grz** extends **S4** (properly) by

(grz) [Grzegorczyk] $\qquad A \dashv \Box A \dashv A \supset A,$

say, (cf. Grzegorczyk [67] or Segeberg [71]). In the latter case, it is interesting to note that **(grz)** is *not* a theorem of Lewis' **S5** (see, e. g., Boolos [79]).

In view of (1G) above, one may try to define Gödel-like translations **H** --> **S4**, $(\ldots)^g$, $(\ldots)^{mt}$ say, matching $(\ldots)^G$, $(\ldots)^{MT}$, resp., but "strengthening" the corresponding clauses for $\Rightarrow_H$ in the direction suggested by (1G). That is:

$$\begin{aligned}
(A)^g &= A, \text{ if } A \text{ is an atom of } \mathbf{H}, \text{ with, in particular,}\\
(\mathbf{f}_H)^g &= \mathbf{F},\\
(A \Rightarrow_H B)^g &= \Box(A)^g \supset (B)^g,\\
(A \wedge_H B)^g &= (A)^g \wedge (B)^g,\\
(A \vee_H B)^g &= \Box(A)^g \vee \Box(B)^g,\\
(\neg_H A)^g &= \Box(A)^g \supset \mathbf{F} \;=\; {\sim}(\Box(A)^g)
\end{aligned}$$

and analogously for the McKinsey-Tarski variant,

$$\begin{aligned}
(A)^{mt} &= \Box A, \text{ if } A \text{ is an atom of } \mathbf{H}, \text{ with, in particular,}\\
(\mathbf{f}_H)^{mt} &= \Box\mathbf{F},\\
(A \Rightarrow_H B)^{mt} &= \Box(A)^{mt} \supset (B)^{mt},\\
(A \wedge_H B)^{mt} &= \Box(A)^{mt} \wedge \Box(B)^{mt},\\
(A \vee_H B)^{mt} &= \Box(A)^{mt} \vee \Box(B)^{mt},\\
(\neg_H A)^{mt} &= \Box((A)^{mt} \supset \mathbf{F}) \;=\; \Box({\sim}(A)^{mt}) \;=\; (A)^{mt} \dashv \mathbf{F}.
\end{aligned}$$

It is likely that the modified translations do still work such as to preserve the corresponding analogue of (G)

$$\text{(g)} \qquad \mathbf{H} \vdash A \iff \mathbf{S4} \vdash (A)^g \iff \mathbf{S4} \vdash (A)^{mt},$$

for any **H**-formula A.

Still, in any one of the cases considered above, **S4** does *not* seem to *explain*, conceptually, anything; it blurs the picture rather than attempting to clarify it.

Consistency of **L!4Q**. Technically, **L!4Q** is *strictly contained* in first-order **S4**. That is: **L!4Q** can be *interpreted* in **S4Q**.

Formulating Lewis' **S4(Q)** *à la* Gödel-Lemmon, with two additional *classical* constants *verum* **T** and *falsum* **F**, say, one has to map atoms into atoms, translating →, &, □ (**of course**), ∀ (linear) by ⊃, ∧, □ (**necessarily**$_4$), ∀ (classical) resp.

Note. It is not very important how **L!4Q**-constants are to be handled. The obvious suggestion is to map ⊤ to **T** and ⊥ to **F**, such as to preserve the translation of negation. Note that, by this kind of translation, ⇒ does *not* go into the ⥽ of **S4**; in fact, A ⥽ .B ⥽ A is *not* an **S4**-theorem; A ⇒ B would rather become □A ⊃ B.

Then **pL**, the **S4**-characteristic-axioms ($\mathbf{K}_{\Box}$), ($\mathbf{i}_{\Box}$), ($\mathbf{4}_{\Box}$) and the exponentiation rule (□) are trivially **S4**-valid. To show that ($\mathbf{k}_{\Box}$) and ($\mathbf{w}_{\Box}$) go also into **S4**-theorems, by such an interpretation, one must realize that

(**k**□) **S4** ⊢ A ⊃ .□B ⊃ A,
(**w**□) **S4** ⊢ A ⊃ (□A ⊃ B) ⊃ .□A ⊃ B.

But, **S4** contains classical logic and (**k**□) is a substitution in a tautology. On the other hand,

(**w**□□) **S4** ⊢ □A ⊃ (□A ⊃ B) ⊃ .□A ⊃ B,

is such a substitution, too, and (**w**□□) yiels already (**w**□), from

S4 ⊢ (□A ⊃ B) ⊃ C ⊃ .(A ⊃ B) ⊃ C

(the latter comes by applying Suffixing twice to the **S4**-axiom □A ⊃ A).

Note. The quantifiers are, again, unproblematic, provided we choose *the right kind of quantified* **S4**.

So **L!4Q** is *consistent*, at least in the same sense first-order **S4** is known to be.

The "modal" *détour* shows that both Weakening and Contraction have been banished only *provisionally* from linear logic. However, the axiomatic formulation is not the best heuristic tool in this respect, it serves to "package" the knowledge, rather than to reveal it, it *hides* the fact that something *else* happened while re-formulating **HQ** in "linear" terms.

The equivalent natural deduction (or type-theoretic) variant of **L!4Q** makes clear the fact that the representation of **H(Q)** in **L!4(Q)** sketched above (after Girard) is *an analysis of relatively complex operations into elementary atomic units*. Moreover, the "linear" decomposition has a good *operational*, "*denotational*" (viz., domain-theoretic) and, in the end, *dynamic* reading.

§6 L!5 is not Lewis' S5. Note that $(5_{[]})$ is, indeed, [Lewis_5], i. e., under the obvious translation, Lewis' characteristic axiom for **S5**. That is: $A^{[]<>} \to A^{[]}$. This yields automatically *consistency* for **L!5**, since $(5_{[]})$ translates into $\Diamond\Box A \supset \Box A$, along the modal interpretation discussed above.

We may *hope* that **L!5** is also "modally interesting", i. e., $(5_{[]})$ does not collapse the ⇒ of **L!5** into *classical* or "material" implication. This caution is motivated by the fact that, unlike for **L!4**,

$$\mathbf{L!5} \vdash A \to .B \to A,$$

whence the *pure implicational fragment of* **L!5** *is at least* **BCK** (the reader could try: "it is *exactly* **BCK**", as an exercise).

So, *if* Contraction, in the form of (**w**) or (**s**), *would be* also available for →, in "un-exponentiated" variant, **L!5** *would* also *contain* the Heyting pure implicational $\mathbf{H}_{\to}$, whence also the full theory of classical ("material") implication ⊃, with, *Peirce's Law* ("un-exponentiated") holding for →, too. Ultimately, **L!5** *would* also contain full classical logic, in view of (Δ) and one *could* be confronted with the unpleasant "equation" **L!5** = **S5**.

Certainly, *if* such a disaster *could ever happen* the "modal" game is not worthwhile playing, in Version Number 5. [Incidentally, however, **R!5** = **S5**.] The fact that this *can not be* the case is shown by the following matrices adapted from Łukasiewicz' three-valued logic. I will re-name the Łukasiewicz "values" **tt**, **uu**, **ff**.

The matrix for ⊥ (*sic*) is **ff**, (so ⊥ is somehow *false* in this world); those for linear implication → and universal exponentiation □ are as follows:

→	tt	uu	ff
tt	**tt**	**uu**	**ff**
uu	**tt**	**tt**	**uu**
ff	**tt**	**tt**	**tt**

	tt	uu	ff
□	**tt**	**ff**	**ff**

From the above, one derives the behavior of −, ◊, □ and ⊠. One has:

□	tt	uu	ff
tt	**tt**	**tt**	**tt**
uu	**tt**	**tt**	**uu**
ff	**tt**	**uu**	**ff**

	tt	uu	ff
−	**ff**	**uu**	**tt**

⊠	tt	uu	ff
tt	tt	uu	ff
uu	uu	ff	ff
ff	ff	ff	ff

	tt	uu	ff
◊	tt	tt	ff

There are many choices for & and ⊕ and the matter is, in a sense, irrelevant for immediate purposes. For the sake of completeness consider also

&	tt	uu	ff
tt	tt	uu	ff
uu	uu	uu	ff
ff	ff	ff	ff

⊕	tt	uu	ff
tt	tt	tt	ff
uu	tt	uu	uu
ff	tt	uu	ff

Note. So, the "non-modal" part works for **pR** [= **LR** or **R** *minus* (**w**)] too, since the matrices above agree, in fact, after renaming, with **Matrix Set XXIV** of John Chidgey in Anderson & Belnap [75] §**29.9**.

Given the finite *algebra* above, it generates *three-valued truth-tables* by "designating" **tt** as Truth. The main observation consists of saying that (**w**) is "not verified", for →, by the "valuation" A = **uu**, B = **ff**, [this must give **uu** as "value" for (**w**)], while ($\mathbf{w}_{\Box}$) is "verified" for the given interpretation of → and □. The reader will check easily that the remaining axioms and rules of **L!5** "hold" for the above "interpretation", as well.

Note. Of course, by conservativity, we cannot collapse **L!5Q** into (*some* form of) quantified **S5**, either.

§7 Catastrophic modal neighbors. Diodorus Cronos and Louis F. Goble are reputed to have asked the question "What would happen if the sun suddenly stopped?" (for Goble, see Anderson & Belnap [75] or even Goble [71]). The answer given in Goble [71] has been anticipated by many modal thinkers, among whom Robert K. Meyer (cf. Meyer [66,68]) and (following Anderson & Belnap) T. C. Mits!

Goble's main answer is of type 4, at best, and he builds up "modally" on **R**. Actually we could have imagined *possibly* weaker (logical) attitudes, as, e. g., building up on **pR**, the proto-relevant logic.

Add, for instance, the following mix-up to this axiomatic proto-world:

[**goble**] ($\mathbf{K}_{\Box}$), ($\mathbf{i}_{\Box}$), ($\mathbf{4}_{\Box}$), (□]), ($\mathbf{k}_{\Box}$).

 [15]

Certainly, $(s_{[]})$ is not worthwhile looking at: it is already in **oR**, and so in **pR** and its extensions. The team called [**goble**] above is, practically, innocent: it worked with **pL**, so why not with **pR**?

The result has been called **R!4**, at the very beginning. This "logic", *relevant* as well, will be, certainly, taken into derision by Girard [on reasons mentioned, e. g., by Nathan Leithes in **La règle du jeu à Paris**, Mouton & Co: The Hague and Paris, MCMLXVII], but seems to have good life-chances, at least philosophically and even proof-theoretically, since (apparently) **it does not contain the bad kind of distributivity** (that is: (d) or (d^-), already mentioned; this is another way of saying that it does *not* contain **R** or that it is *not* the same thing as Goble's logic of **type 4**).

Of course, the would-be **R!5** is the same as Lewis' **S5**, as argued above and as shown by Goble, long before [71].

This completes a raw approximation of the modal story. In a sense, Number 5 is the Ultimate Limit, since "exponentiating" on **pL**, *à la* **Sk**, with $5 < k \leq 9$ is (following Goble, again) even beyond everything that "only God knows"!

S8 Why L!5 (i. e., yet another "linear" logic)? As mentioned earlier, Girard's translation of Heyting's logic **HQ** into **L!4Q** is, in fact, a way of *analyzing* **HQ** into linear operations.

Given Gödel's *double negation interpretation* $(..)^{NN}$ of classical logic into first-order Heyting, (roughly speaking, this sends classical atoms p into double intuitionistic negations $\neg\neg p$, and makes the resulting extension "commute", as a map, with the connectives), one may try to transfer Girard's translation to the classical case, by composing $(...)^{L}$, defined previously, with $(...)^{NN}$.

Ignoring quantifiers again, the reader will check easily that the result leads, in substance, to the following awkward "representation" of the classical connectives into **L!4** (now $\supset$, $\wedge$, $\vee$ and $\sim$ are *defined* as linear **L!4**-"connectives"):

$$
\begin{array}{lll}
[Df\supset] & A \supset B & := A^{\langle\rangle[]} \rightarrow B^{[]\langle\rangle} \\
[Df\wedge] & A \wedge B & := (A^{\langle\rangle[]} \boxtimes B^{[]\langle\rangle[]})^{\langle\rangle} \\
[Df\vee] & A \vee B & := A^{[]\langle\rangle} \Box B^{[]\langle\rangle} \\
[Df\sim] & \sim A & := A^{-\langle\rangle} = (A \rightarrow \bot)^{\langle\rangle}.
\end{array}
$$

This looks rather obscure *qua* "linear analysis of classical logic". Of course, the correct way of expressing the meaning of the above is to define inductively (on the structure of classical formulas) a translation, $(...)^{CL} : \mathbf{C} \longrightarrow$ **L!4**, say, (cf. Girard [87]), by:

$$
\begin{array}{llll}
(A)^{CL} & = (A)^{[]\langle\rangle}, & & \text{if } A \text{ is an atom of } \mathbf{C}, \\
(A \supset B)^{CL} & = ((A)^{CL})^{[]} \rightarrow (B)^{CL} & = & (A)^{CL} \Rrightarrow (B)^{CL}, \\
(A \wedge B)^{CL} & = (((A)^{CL})^{[]} \boxtimes ((B)^{CL})^{[]})^{\langle\rangle} & = & ((A)^{CL} \& (B)^{CL})^{[]\langle\rangle}, \\
(A \vee B)^{CL} & = (A)^{CL} \Box (B)^{CL}, & & \\
(\sim A)^{CL} & = ((A)^{CL})^{-\langle\rangle} & = & ((A)^{CL} \rightarrow \bot)^{\langle\rangle}.
\end{array}
$$

The above shows, at best, that classical logic **C** is a strange mixture of disparate ideas, but, ultimately, is not very entertaining, either.

The alternative consists of replacing *Girard's translation* $(...)^{L}$ by a mapping **C** $\longrightarrow$ **L!5** (sic). For convenience, the change of domain and range will be reflected in notation by having an "L" *sub*-scripted in place of a *super*-scripted one. Thus: $(...)_{L}$.

If taken as definitions in **L!5**, [Df⇛], [Df∧] and [Df~] give the corresponding $(...)_{L}$: **C** $\longrightarrow$ **L!5**. Indeed, re-using $⇛_C$, $\wedge_C$ and $\neg_C$, this time for the official *classical* connectives ⊃, ∧, ~, resp., one has, inductively again:

$$\begin{array}{llll}
(A)_L & = A, & & \text{if } A \text{ is an atom of } \mathbf{C}, \\
(A ⇛_C B)_L & = ((A)_L)^{[\,]} \to (B)_L & = & (A)_L ⇛ (B)_L, \\
(A \wedge_C B)_L & = (A)_L \,\&\, (B)_L & = & (A)_L \wedge (B)_L, \\
(\neg_C A)_L & = ((A)_L)^{[\,]} \to \bot & = & (A)_L ⇛ \bot.
\end{array}$$

[The extension to the first-order case must be "commuting", too, in the obvious way.]

Note. For classical *disjunction* we have several alternatives. However, applying *à la lettre* Girard's suggestion for **H**:

$$(A \vee_C B)_L = ((A)_L)^{[\,]} \oplus ((B)_L)^{[\,]} \;=\; (A)_L \vee (B)_L,$$

to the classical case does not yield a proof-theoretically well-behaved disjunction as *intended* in a classical setting. We could have defined, instead, in **L!5(Q)**:

[Df∨]	A ∨ B	:= (A~ & B~)~	(*global* [*Ockham*] **or**, *external* **or**),
[Df⊔]	A ⊔ B	:= □A □ □B	(*global fission*, *internal* **or**),
[Df+]	A + B	:= (A ⇛ B ⇛ B)	(*global inferential* **or**),

whence the *intended classical* **or** must be the one given by [Df∨].

The reader will eventually check that, for all **H**-formulas A and all **C**-formulas B,

$$\mathbf{H} \vdash A \implies \mathbf{L!4} \vdash (A)^{L} \quad \text{and} \quad \mathbf{C} \vdash B \implies \mathbf{L!5} \vdash (B)_{L},$$

and similarly for the first-order analogues. So, if **L!4(Q)** is a correct way of explaining Heyting's logic, one may hope that **L!5(Q)** is an equally good *medium of linearization*[10] for *classical* logic.

Draft: Paris, June 1987.
Revised: Nijmegen, July 1987, 11 August, 7 September 1988.
Last revised: Nijmegen, 28 August 1990.

Notes.

* **Acknowledgment.** I am indebted to Jean-Yves Girard for *providing reasons* to write this up. The ***axiomatics for linear logic*** arose first as a *theory of linear combinators*, after one of his talks. **L!5**(Q) [= *Ockham logic*, cf. *Summa logicae*, *Pars Secunda et Tertiae Prima*, ed. Ph. Boehner] is ***orthogonal*** to the observation that "*la logique linéaire, c'est - quand-même - une logique de Girard*". William Ockham has been oft credited with **S5**(Q), but I think that even **L!5Q** is - after all - *a logic of Ockham*.

[1] This is *not* immediately obvious, since Girard's original syntactic decisions are rather *unheimlich* to the unwarned reader. *Provability* is introduced in Girard [87] by an artifice claimed to be a "Gentzen sequent calculus"; the latter is quite unusual, and might be rather called "sequent axiomatics", provided "sequents" are understood appropriately. The syntax appearing here *diverges* from that of Girard in several respects. This has been mainly indended in order to facilitate comparisons with neighboring or rival logics.
(1°) Unlike here, Girard has also *negative atoms* $p^{\perp}$, $q^{\perp}$, ..., as primitives. Next, he introduces "linear negation" by an inductive re-writing scheme on negation-free formulas; this means that neither his "negation" nor anything else defined in terms of it (e. g., even linear entailment) are really "logical connectives" in the ordinary sense.
(2°) Girard takes the four "propositional constants" **T**, **⊥**, **t** and f and ▯, ⊠, &, ⊕, as *primitive*.
(3°) Also, my *terminology* diverges from Girard's in that I use "linear fusion", "linear fission" in place of "linear **and**", "linear **or**"; this way of speaking is derived from work on relevant logics (Alan R. Anderson, Nuel Belnap, Robert K. Meyer *et al.*) and has a perfect motivation: the addition of Contraction to the fragment called "proto-linear" here, gives a "proto-relevant logic", with connectives "fusion" and "fission" defined as above (this is, in fact, exactly **LR**, or "non-distributive **R**" of Thistlewaite *et al.* [87], mentioned later).
(4°) Finally, the remaining disagreements are *typographical*: Girard's **par**, (denoting my "linear fission" and) printed as a reversed ampersand ⅋, is replaced here by what a computer scientist has always called "par" and printed as ▯ (in slightly different a context, indeed, following Dijkstra *et alii*). If functioning as "negation", Girard's superscripted "orthogonality" **⊥**, is - more comfortably - printed here as a superscripted bar, and his "linear entailment", **⊸** has now become an arrow →. And, since I said that Girard's **!** and **?** are "modalities", without denying the fact that they are also "exponentials", I will print them as *super-scripted* ▯ and ◊, reserving (mainly) ! for a better usage, in a larger context.

[2] Due to so many differences, the most hygienic strategy to show that **L!4**(Q) and Girard's "linear" logic are *the same* thing, would probably consist of showing that **L!4**(Q) is (sound and) *complete* for Urquhart's *monoidal semantics*, as extended

by Girard [87], and called, for some reason, "phase semantics", there. An easier, although slightly abusive, strategy, (given the presence of "negative" atoms) consists of realizing first that his "right-handed-gentzenization" is, in fact, a disguised axiom system. By the same token, one should note, as well, that his "CUT" covers both [*Modus ponens*] and the Transitivity of linear entailment, i. e. the rule

(trans) A → B, B → C ==> A → C.

The reciprocal derivability of theorems becomes then trivial. The interested reader can recover an alternative proof of equivalence, by inspecting Avron [*87] (which came to my attention after completing a first draft of this paper; Avron's axiomatics - for **L!4** only - are based on different insights and are less useful for a proper *combinatory analysis* of **L!4Q**).

3 But my "4" in **L!4(Q)** is rather reminiscent of Lewis' **S4** and has nothing to do with Smiley's "4", which is part of a local, *ad hoc* nomenclature (e. g., Smiley's $\mathbf{L}_1$ is the pure implicational part of Hilbert's "positive logic").

4 The same idea has been exploited by Urquhart in various directions by specializing the underlying monoids: the semi-lattice option leads to models for *relevant logics*, while more "algebraic" specializations lead to models for **BCK**-extensions, and, in general, for **BCK**-based logics *without contraction*. For instance, taking *ordered abelian groups* as a starting point, one obtains models for Łukasiewicz' many-valued system Łω, this approach has been proposed earlier (1970), by Dana Scott (see Urquhart [86] for details).

5 Girard's distinction between "multiplicative" and "additive" is visible and convincing only in Gentzen-style *Sequenz*-formulations and refers to the way of composing "contexts" in the conclusion of a derivation rule in different variants of such formulations. Since Gentzen *Sequenz*-systems are here only of a remote interest, I will *not* use this way of speaking any further. See Girard's lecture notes [*87] for an elementary explanation.

6 Girard has denied this (in print and otherwise), but his reasons were not among the best possible ones.

7 So it is "relevant", whatever means this *philosophically*. Proof- (or better type-) theoretically, this claim amounts to the fact that "cancelling" typed combinators (= closed typed λ-terms) are *not* admitted as objects.

8 This makes it "linear", in a rather precise sense.

9 Or rather *minimal*, in the sense of Johansson.

10 [Added in proof: August 1988]. The hope is *justified* and leads eventually to *a local proof-theory for classical logic*. The details will appear elsewhere [Rezus 9*].

R E F E R E N C E S

Anderson, A. R., N. D. Belnap Jr. *et alii*
75 **Entailment** I, Princeton University Press, Princeton, 542 pp.
8* **Entailment II**, Princeton University Press, Princeton [in preparation].

Avron, A.
*87 *The semantics and proof-theory of linear logic*, University of Edinburgh, Laboratory for Foundations of Computer Science Report Series, **ECS-LFCS-87-27**, April 1987, 29 pp. [also as: **CSR-232-87**; now published in: **TCS 57**, 1988, pp. 161-184].

Boolos, G.
79 **The Unprovability of Consistency**, Cambridge University Press: Cambridge (UK), 184 pp.

Church, A.
51 *The theory of weak implication*, in: A. Menne, A. Wilhelmy and H. Angstl (eds.) **Kontrolliertes Denken**, Kommissionsverlag Karl Alber: Munich, pp. 22-37.
51a *Minimal logic* (abstract), **The Journal of Symbolic Logic 16**, page 239.

Dunn, J. M.
66 **The Algebra of Intensional Logics**, Dissertation, University of Pittsburgh, Pittsburgh PA, 177 pp.
86 *Relevance logic and entailment*, in: Gabbay & Guenthner [86], pp. 117-224.

Gabbay, D. M and F. Guenthner (eds.)
86 **Handbook of Philosophical Logic III**, D. Reidel Publishing Company: Dordrecht, Boston, etc.

Girard, J.-Y.
87 *Linear logic*, **Theoretical Computer Science 50**, pp. 1-101.
*87 **Lambda-calcul typé**, *Cours de DEA*, Université de Paris VII, [lecture notes from 1986/87; revised as **Proofs and Types**, Cambridge University Press: Cambridge UK, etc., 1989].

Goble, L. F.
71 *A system of modality*, **Notre Dame Journal of Symbolic Logic 12**, pp. 225-237.

Gödel, K.
33 *An interpretation of the intuitionistic propositional calculus*, = [1933f] in: Gödel [86], pp. 300-302.
86 **Collected Works I**, Oxford University Press and Clarendon Press: Oxford and New York.

Grzegorczyk, A.
67 *Some relational systems and the associated topological spaces*, **Fundamenta Mathematicae 60**, pp. 223-231.

Hacking, J.
63 *What is strict implication?*, **The Journal of Symbolic Logic 28**, pp. 51-71.

Jaśkowski, S.
63 *Uber Tautologien in welchen keine Variable mehr als zweimal vorkommt*, **Zeitschrift für mathematische Logik und Grundlagen der Mathematik 9**, pp. 219-228.

Łukasiewicz, J.
30 *Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenskalküls*, **Comptes Rendus des séances de la société des sciences et des lettres de Varsovie**, Classe III, **23**, pp. 51-77.

McCall, S.
66 *Connexive implication*, **The Journal of Symbolic Logic 31**, pp. 415-433.

McKinsey, J. C. C. and A. Tarski
48 *Some theorems about the sentential calculi of Lewis and Heyting*, **The Journal of Symbolic Logic 13**, pp. 1-15.

Meredith, C. A and A. N. Prior
63 *Notes on the axiomatics of the propositional calculus*, **Notre Dame Journal of Symbolic Logic 4**, pp. 171-187.

Meyer, R. K.
66 **Topics in Modal and Many-valued Logic**, Ph. D. Dissertation, University of Pittsburgh, Pittsburgh PA.
68 *Entailment and relevant implication*, **Logique et analyse 11**, pp. 472-479.

Moh Shaw-Kwei
50 *The deduction theorem and two new logical systems*, **Methodos 2**, pp. 56-75.

Priest, G. and R. Routley [= Sylvan]
*83 **On Paraconsistency**, *Research Papers in Logic 13*, Research School of Social Sciences, The Australian National University, Canberra, 231 pp.

Priest, G. and R. Routley [= Sylvan] (eds.)
84 **Paraconsistent Logics**, issue of **Studia Logica XLIII**, 1/2.

Priest, G., R. Routley [= Sylvan] and J. Norman (eds.)
8* **Paraconsistent Logic**, Philosophia Verlag: Munich [to appear].

Prior, A. N.
62 **Formal Logic**, Clarendon Press: Oxford [second edition].

Rezus, A.
82 *On a theorem of Tarski*, **Libertas Mathematica** (Arlington TX) **2**, pp. 63-97.
9* **Lambda Calculus Methods in Proof Theory**, [to appear].

Routley [= Sylvan], R., R. K. Meyer, V. Plumwood, and R. T. Brady
82 **Relevant Logics and Their Rivals I**, Ridgeview Publishing Company: Atascadero CA, [POB 686, Atascadero CA 93423], 460 pp.

Segeberg, K.
71 **An Essay in Classical Modal Logic**, 3 volumes, University of Uppsala: Uppsala.

Smiley, T. J.
55 **Natural Systems of Logic**, Doctoral Dissertation, University of Cambridge (UK), 184 pp.
58/9 *Entailment and deducibility*, **Proceedings of the Aristotelian Society**, (n.s.), **59**, pp. 233-254.

Thistlewaite, P. B., M. A. McRobbie, and R. K. Meyer
87 **Automated Theorem-Proving in Non-Classical Logics**, Pitman: London and Wiley: New York, 1987.

Troelstra, A.S
86 *Introductory note* to Gödel [33], in: Gödel [86], pp 296-299.

Urquhart, A.
72 *Semantics for relevant logics*, **The Journal of Symbolic Logic 37**, pp. 159-169.
72a *A general theory of implications* (abstract), **The Journal of Symbolic Logic 37**, page 443.
72b *The completeness of weak implication*, **Theoria** (Lund) **37**, pp. 274-282.
73 **The Semantics of Entailment**, Ph. D. Dissertation, University of Pittsburgh, Pittsburgh PA, 66 pp.
74 *Proofs, snakes and ladders*, **Dialogue** (Canada) **13**, pp. 723-731.
86 *Many-valued logic*, in: Gabbay & Guenthner [86], pp. 71-116.