IM BUCHSTABENPARADIES

Gottlob Frege and His Regellogik

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équivalences
NIJMEGEN
The Netherlands
2016

for ALICE, of BLUE VEGAS

1 The Many Names of *The Truth*

non idem est si duo dicunt idem
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The Originary Symbolic Dichotomy (*Zeichenarten*)

On the first page of his **Begriffsschrift**[2], Gottlob Frege begins the "explanation of designations" (*Erklärung der Bezeichnungen*) appearing in his Concept[ual] Script (*Begriffsschrift*)[3] with a *distinguo* he estimates to be a "fundamental idea" (*Grundgedanke*) and claims that he intends to make it useful for the Domain – or Realm – of Pure Thinking ([*das*] *Gebiet des reinen Denkens*).

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[1]This is the way most Romanians – even high school teenagers – would eventually quote the Latin saying, following a poem of their national poet, Mihai Eminescu: "Noi amândoi avem acelaşi dascăl, // Şcolari suntem aceleiaşi păreri... // Unitul gând oricine recunoască-l. // Ce ştii tu azi, eu am ştiut de ieri. // De-aceleaşi lucruri plângem noi şi râdem... // *Non idem est si duo dicunt idem*.", etc. [We, both of us, have got the very same old teacher, // Both you and me are pupils of the same opinion hence... // We're one-in-thought, let this be known to everybody ever. // Nevertheless, what you today know was my yester knowledge once. // Together crying, laughing at same things, and both in tandem... // Well, friend of mine: *non idem est si duo dicunt idem*. (My translation. AR)] — *Pace* the poetical inversion, Eminescu is, actually, misreading Terence, here, together with a later learned – likely Jesuit – paremiological tradition: '*duo cum faciunt idem, non est idem*', '*si duo faciunt idem, non idem est*', etc. (The Jesuits were thereby equipping a former sclave with subtle – their own – political views!) Cf. Publius Terentius Afer, **Adelphoi**, V.3, 821–825: MICIO *multa in homine*, DEMEA, *signa insunt, quibus ex coniectura facile fit, duo quom idem faciunt, saepe ut possis dicere 'hoc licet inpune facere huic, illi non licet', non quo dissimilis res sit, sed quo is qui facit*. In more recent times, the Latin saying had '*dicere*' for '*facere*'' ('*si duo dicunt idem, non idem est*') and classics scholars would usually render the modified variant by 'If two *languages* say the same thing, it is not the same thing.'. So, unlike in the Jesuit reading, the modified Latin saying was about [two] *languages*, not about *people* (as in Eminescu's poem and as currently understood and / or oft quoted, colloquially, by most Romanians)! See, e.g., Jon R. Stone **The Routledge Book of World Proverbs**, Routledge 2006, etc.

[2]**Begriffsschrift**, *Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle a/S, Verlag Louis Nebert 1879; henceforth referred to as BS, followed by paragraph and page number, in this edition. Unless specified explicitly otherwise, I will use my own translations, they are usually free renderings of the German original. As a rule, German terms are mentioned, in parentheses, only the first time an English translation occurs in the text. – There are two English translations of BS: **Concept Script**, [translated by] Stefan Bauer-Mengelberg, in: Jean Van Heijenoort (ed.), **From Frege to Gödel:** *A Source Book in Mathematical Logic, 1879–193*, Harvard University Press, Harvard MA 1976, and **Conceptual Notation and Related Articles**, with a biography and introduction, [translated by] Terrell Ward Bynum, Clarendon Press, Oxford UK and Oxford University Press, New York NY 1972 [Oxford Scholarly Classics].

[3]As well as in BS, the **Begriffsschrift** (the book) itself.

Paying attention to his colloquialisms, as well as to the formal terminology, the basic *distinguo* states a kind of symbolic proto-dichotomy *Ur-zwiespalt* at the root of the mathematics[4], and runs as follows:

There are only two kinds (*Arten*) of symbols (*Zeichen*) in mathematics: the first kind consists of letters (*Buchstaben*) serving to express generality (*Allgemeinheit*), the second one consists of symbols that have a definite sense ([*solche*] *die einen ganz bestimmten Sinn haben*).

From his examples and further explanations, we are first inclined to render the distinction, in modern terms, as one between *variables* and *constants*. This is not exactly the case.

The first kind contains *Buchstaben* only, and covers actually what the moderns would call *variables*, as well as *meta-variables* (perhaps even *meta-meta-variables*!).

The second kind covers what we mean nowadays by *constants*, in a large sense, though. Some of Frege's "designations" would *translate* to our "constants", indeed (i.e., they would become "constants" – in our sense – after translation). Yet, we don't have exact equivalents for all of them. In the end, this shouldn't cause serious problems, because we can always invent some notation, in order to provide would-be equivalents.

The real trouble appears while attempting to translate the first kind of symbols (i.e., Frege's "letters") into modern terms. This doesn't work smoothly.

Unlike most of our logic fellows – among the *recentiores*, those of the last fifty years, say –, Frege has a veritable *Buchstabenparadies*:

- Greek, Latin, and German letters (for short: Grk, Lat, Ger),
- the *case distinction* matters, for him, too: *uppercase*, *lowercase* (UC, LC), and, sometimes, even
- the distinction between *vowels* and *consonants* (V, C) is going to be relevant.

So we have combined syntactic types: Grk-UC-C, Grk-LC-V, Lat-UC, Lat-LC, Ger-LC-V, etc. We might call them *sub-kinds*, for instance. Fortunately, not all combinations[5] would be actually used. For instance, the V/C distinction won't really matter in Latin type.

There is more, however.

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[4] *Grössenlehre* in BS, but this covers, actually, nearly all of what would have counted as "mathematics" in his times.

[5] Here $18 = 12 + 6$, because one can distinguish among distinctions, too, so that we can have, in principle, Grk-UC-C, and Grk-UC-L, on a par with Grk-UC, all of them as distinct sub-kinds, after all.

If, as regards the *vowels*, only two sub-kinds are reserved for specific purposes, viz. Grk-LC-V, and Ger-LC-V, some of the *consonants* are reserved for some uses, while some other are meant for different uses.

The overall design – of the sub-kinds in *Buchstaben* – is governed by what we might call "semantic criteria" and / or, even, "ontological presuppositions".

In order to understand the full meaning of Frege's Conceptual Script we must revert to a later work, namely to his **Grundgesetze der Arithmetik**[6]. This work makes use of the mature Fregean distinctions: *Sense vs Denotation* (*Sinn / Bedeutung*), *Object vs Function* (*Gegenstand / Function*), and *Function vs Concept* (*Function / Begriff*)[7].

Ontology first

Frege's ontological equipment is rather minimalistic: there are only two categories of entities in the world ("out there"): Objects and Functions. Ultimately, the Senses (of the words) are "out there", too, because they seem to be objective. Yet, inasfar the notation itself (i.e., the Conceptual Script) is concerned, only the former two would actually matter.

The Ojects are *saturated* (*gesättigt*) or *complete* – self-sufficient to themselves[8] –, while the Functions are not so.

There are, in particular, two very special Objects "out there", entities Self-sufficient to Themselves as every Object, endowed with a Biblical status, more or less, called "*das Wahre*" and "*das Falsche*", in Frege — *The Truth* and *The Falshehood*, in

[6] **Grundgesetze der Arithmetik**, *begriffsschriftlich abgeleitet*, I. Band, Jena, Verlag von Hermann Pohle 1893; II. Band, 1903; [reprinted by Hildesheim, Olms Verlag 1962], henceforth GGA:1, GGA:2. As a matter of fact, only the first volume would matter for present purposes. The previous remarks on translations for BS apply to GGA, too. — GGA has been partially translated into English in: **The Basic Laws of Arithmetic**, [translated by] Montgomery Furth, University of California Press, Berkeley and Los Angeles CA 1964. There is also a ("very rough") translation of GGA:2 (§§ 53 to the end), by Richard G. Heck [Jr.] and Jason Stanley, 2004 (online at Brown University, Providence RI), as well as a group at *Arché*, St Adrews GB, working on a complete translation, to be likely published by Oxford University Press. [Added in print, March 15, 2016. A complete English translation of GGA has appeared in 2013: **Basic Laws of Arithmetic.** *Derived using concept-script*, Volumes I & II (in one), Translated and edited by Philip A. Ebert & Marcus Rossberg, with Crispin Wright, Oxford University Press, Oxford 2013.]

[7] Frege's *Function* is not a "function" in our sense – not even in the sense of his mathematical contemporaries –, so I might be tempted to leave *Function* untranslated. However, in order to make things look uniform in Middle English, I would always capitalize Frege's Objects, and Concepts, on a par with his *Functionen*, and equip the latter with an English plural, so we can read colloquially, anyway.

[8] The Mediaeval's *Nihil* (✠, say), the Arab's Zero (0), and the Modern's Empty Set ($\emptyset$), – even the Unicorns ($\varnothing$, ◯, and ⊙) – are best examples in point. This is, actually, Frege's Paradigm of an (Individual) Object, because Numbers, People, The Earth, The Moon, The Sun, The Stars, and whatever else admitting of a concrete or abstract (mind-) pointing to... (*Meinen*) are democratically equal first-level Citizens of his World, and are going to have exactly the same ontological status as 0, $\emptyset$, $\varnothing$, ◯, etc.

Middle English, or *verum* and *falsum*, in Mediaeval Latin. We shall reserve special (Proper) Names for the latter: $\top$ and $\bot$, for instance[9].

By Originary Ontological Dichotomy, the Functions are seen (and declared) to be *ungesättigt*, *non-saturated*, *incomplete*, and *in need of completion* or "hungry" (*ergänzungsbedürftig*), so to speak: they are always ready to eat something (else). Eventually, it would turn out they are able to eat nearly everything, not only Objects, but also Functions, so that Self-Eating, in guise of Function-Autophagy, is tolerantly allowed in the Script.

The Concepts are special cases of Functions, they are *pure and simple* or *simply hungry* (*einfach ergänzungsbedürftig* – the due meta-conceptual explanation comes in a moment), so they are non-saturated, and in need of completion, too[10].

Names, Fathers, and Sons

Like in the Bible – the Old Testament –, more or less, everything can be *named* in Frege's Script[11]. Beware, though: there are Names and Names in Paradise!

The Names of Objects are called *Proper Names* (*Eigennamen*).

Some of them are granted to us, as e.g., the Name of Socrates, the Name of 0, the Name of 1 (the latter are *Ziffern*, of course), and, in order to distinguish between the Name of Rose (*sc.* the sister of Miss White – actually Mrs Black[12]) and Rose herself, or else between Lady Di[13] and Her Name, we must use quotes[14]. Single

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[9]These Names are *not used* in the Script, however. They are just Meta-Names, *we use* (in order) to speak about those Very Special Objects, while speaking about Frege, and his (Conceptual) Script, for instance.

[10]This being said, one can realize, after a while, that Frege was a Left Wing Logical Radical, in fact, because The Hungry Ones – traditionally, a factor of instability in the World – are likely going to eat nearly everything eventually – well, mainly Objects, but also a bit of themselves – in order to make The Realm of Pure Thinking (*das Gebiet des reinen Denkens*) possible.

[11]Including The Truth Itself (*das Wahre*), and Its Eternal Foes, The False [One] (*das Falsche*), in particular. Except, perhaps, for Everything (Itself). There is no Name for Everything – for The Overall Togetherness, I mean –, in Frege. At least, this was the initial intention. And this, probably because, otherwise, The Name of Everything would have been a Name of Itself, thereby including Itself, as well as The Name of Its Name, The Name of All Its Names, and so on. — In fact, as (the later) Lord, Sir Bertrand (Third Earl) Russell was able to show a bit later, Everything *can* be named in the Script Itself (as well as and in GGA), because every Fregean (Proper) Name is a Name for everything, and this is already a Theorem of The Script Itself. Let's not anticipate, however.

[12]See the *Appendix*. In the meantime, the former Miss Rose White married Mr Black, the topologist.

[13]My cat (see below).

[14]They are not exactly the usual German *Gänsefüßchen*, in Frege; and they do *not* ressemble the French *guillemets* nor the English double / single quotes, either. The Fregean quotes might be called "German single quotes" (opening-down and closing-up), but the modern German typography doesn't seem to use them anymore. In what follows, I will tacitly translate the Fregean quotes into (contemporary) British & American English *single quotes* (opening-up and closing-up), like in 'Alice', for instance, the Name of Miss White, the twin-sister of Mrs Black.

quotes, to be precise. So, as expected, the Name of Rose should be 'Rose'[15], while the Name of The Cat – my cat, the one sleeping *next to* The Mirror, as well as *in* The Mirror –, should be 'Diana'[16], even for Gottlob Frege[17].

All this looks, *prima facie*, childish. Incidentally, however, Frege detects a bad use of the Proper Names in the mathematical writings of his very learned contempporary fellows (confusions between *Ziffer* and *Nummer*, for instance) and takes the opportunity to poke some fan on the subject. Consequently, in order to be sure that my very learned contemporary fellow readers won't confuse those many Bostons, out there (in the World) or in here (on Paper / in the Mirror), the Name of 1 (read *Eins* in German) is '1', not 1. To make a long story short, *the numeral* is the name of *the number*, while the number itself is not – and cannot be – a name, even if we are gödelizing **Principia Mathematica**[18].

The remaining Fregean Proper Names can be manufactured easily, by using the definite article[s] (and, perhaps, the demonstratives), like in the case of the Names of the Planet Venus[19], 'The Father of Socrates', 'The Square Root of (*Quadratwurzel aus*) 4', and so on.

This is important in what follows, because 'The' (*der / die / das usw*, and its modern translations, i.e., a Name of The Definite Article, no matter in what modern language, be it German, English, Dutch, Romanian, or Arabic / MSA) is a primitive symbol in GGA. Yet, beware! 'The' is not a Proper Name.

In modern terms, the latter kind of Fregean Proper Names – the ones we can make with 'The' – are called "*descriptions*"; to Frege they are Proper Names, as well.

Now, all Fregean Proper Names are Names of Objects, even if obtained by composition, like 'The Father of Socrates', for instance.

[15] And, possibly, 'Rose Black', according to her passport.

[16] Diana (The Cat, I mean) has gotten a passport of her own, and that's Her Name, indeed, according to the Appropriate Minister of Her Majesty, The Queen. [Added in print (March 15, 2016). We got a King now, but I wrote the paper in 2009.]

[17] The modern typographical alternative (TEX, LATEX, AMSTEX, etc.), using *quasi-quotes* – a recent invention of Professor Willard van Orman Quine –, was not available to *Herr* Hermann Pohle, Frege's Publisher of Jena. We shall, therefore, refrain, in general, of putting The Cat, Lady Di, or other Distinguished Objects between quasi-quotes, like in $\ulcorner$Diana$\urcorner$.

[18] Some of the very learned characters appearing in Jonathan Swift's **Travels** would eventually use Objects in place of Names in order to discuss philosophical questions, but this is a slightly different epistemic scenario: those brave people were, by no means, confused. (Imagine, for a moment, the headache – and all that suffering – it would have taken to do [scientific] semantics or model theory! Or, even, proof theory: no way to draw those nice [Gentzen] trees, nor any kind of pictures, anymore!) On the other hand, if some of my learned readers would think that such confusions are unlikely in modern / contemporary mathematics, I should strongly dissent. Enough to open a recent book on categories, for instance, in order to get the fun.

[19] 'The Morning Star', 'The Evening Star', etc. Plenty of XX-th century philosophers have fallen in love with Her (Its?) Names!

The Fregean technical jargon is, in this case, as follows: The Father of Socrates (i.e., Sophroniskos) is the *denotation* (*Bedeutung*) of (The Proper Name) 'The Father of Socrates', and, likewise, Socrates himself is the denotation of 'Socrates' etc.[20].

For Venus and her (its?) philosophically beloved Names we have yet another useful *distinguo*: The Proper Names of Venus have the planet itself (herself?) as a denotation; yet all those Names are different, and their differ by their Senses.

So 'The Morning Star' and 'The Evening Star' *express* different Senses, even if they denote / stand for the same Object (The Planet Venus).

And, *mutatis mutandis*, we get the same story, about '5' and '2+3', for instance.

If we extract, now, The Name of Socrates from 'The Father of Socrates' (a Proper Name), we don't get a Proper Name anymore, but a Name of something else, something in need for completion, viz. the Name of an un-saturated entity, i.e., a Function-Name, in Frege's terminology; this one would be 'the Father of...', a Name that *indicates* (refers to, *deutet an*) a "hungry" thing (it is a Function, and, in particular, a Concept: see below). In this case, Frege would have said that the Object Socrates *falls under* (*fällt unter*) the Concept referred to by the Function-Name 'the Father of...'[21].

Whence, "Falling Under" would be a relation – in our terms, and for Frege too – subsisting between an Object and a Concept. We shall come back to this important Relation later.

What about something like: 'Sophroniskos is the Father of Socrates'?

Obviously, we can play the extraction trick, in this case, too: if we extract The (Proper) Name of the Father, we get something looking, more or less, like what we had before, viz. '...[is] the Father of Socrates' – yet another Function-Name[22].

But there is more to do, here: we can also extract The Name of the Son from what remained, getting this time '...[is] the Father of...'. According to the convention on

[20] In other words, The Cat is the denotation of Her Name: easy to remember! Of course, on this plan, Lady Di is also the denotation of The Name of The Cat, but this is only because Alice – Miss White – use to call my cat a Lady.

[21] Unlike for most of the moderns, like Freud, for instance – who was able to draw the otherwise unlikely picture of a Self-Satisfied and Self-Sufficient Father, Master of Everything, Eating Himself, etc. –, the Fathers are always hungry for Frege. As a rule, they are eating just Sons and Daughters, and – in Truth –, only their own. Because, again – more or less like in the Bible, the New Testament, this time –, One can possibly be – in Truth – only The Father of His Own Son. There is an obvious progress, however, mainly in the direction of the recent Feminist Movements: in Frege, One can have True Daughters of His Own, as well. In the end, the implied Parental Relation is equally patriarchal (a Falling submissively Under), but one should note the fact that the Fregean Mothers are, actually, in the same position of (Conceptual) supremacy.

[22] This is, for sure, a Concept-Name, too, pointing out to (*deutend an*, or referring to) the Concept of a specific Father, the Father of that Brave Old, Wise Man, vividly pictured in the dialogues of Plato, and Xenophon, mocked in the plays of Aristophanes, and so on.

saturation / incompleteness, this should be The Name of a Function, too; namely of one that is *twice hungry* (*zwiefach ergänzungsbedürftig*), so to speak.

Now we can say what kind of Function is a Concept: a Concept is an unsaturated entity that is *hungry simpliciter* (*einfach ergänzungsbedürftig*), *once hungry*: Concepts can eat only once, to saturation.

To simplify this kind of (meta-meta-) talk, Frege introduces the generic means to express *generality* we encountered at the very first page of BS: *die Buchstaben*.

Willkommen im Paradies

Required to this purpose is the sub-kind said Grk-LC-C, before. The corresponding letters are used to fill in the holes marked "..." in the above.

Actually, there are only two of them, ξ and ζ: the first one, ξ, is used to fill in the single hole of a Concept-Name, while both ξ and ζ are used to fill in – in this order – the two holes occurring in a Function-Name that is "twice-hungry"!

In either case, both ξ and ζ are used only in guise of *place-holders*, and they are called "arguments". As usually in mathematics, in fact. The coresponding holes are said to be *argument-places* or *argument-positions* (*Argumentsstellen*).

So we have Concepts or One-Argument Functions, and Two-Argument Functions, so far.

The Functions have Names of their own, namely *Function-Names* (*Functionnamen*). Like what they name, the Function-Names are hungry, and in need for completion, they are *Incomplete Names*, or Names with Holes, as in: 'the Father of...', or in '...[is] the Father of Socrates' (Names of two different Concepts), and '...the Father of...' (a would-be Name of The Relation Subsisting Between Father and Son), resp.

In both cases, the Holes are mere Places or Locations for Arguments (*Argumentsstellen*) and they should disappear in favor of the corresponding Place-Holders, Greek Letters in sub-kind Grk-LC-C: ξ, in the first case, or ξ and ζ – in this order – in the second case.

Now, as we might want to get rid of all those boring examples (with Fathers and Sons, resp. Sophroniskos and Socrates, in particular), and speak in general – *in the Name of Generality*, so to speak –, we can also add yet another sub-kind of *Buchstaben*, with, this time, an additional Object / Function *distinguo*.

The additional distinction O / F generates sub-sub-kinds: Grk-UC-C-O and Grk-UC-C-F, say.

To this purpose, four letters in sub-kind Grk-UC-C – e.g., Γ, Δ, and Φ, Ψ –, two parentheses, and an inevitable comma[23], should largely suffice, for a while.

- Grk-UC-C-O for Object meta-variables: Γ, Δ, instead of 'Socrates', 'Sophroniskos', and the like,
- Grk-UC-C-F for Concept meta-avariables: Φ, instead of 'the Father of...', or '... [is] the Father of Socrates',
- Grk-UC-C-F for [Two-Argument] Function meta-avariables: Ψ, instead of '...[is] the Father of...'.

Remember: all of them – *die Buchstaben* thus – are "means to express generality", as said before.

So we get now things like

- Δ, Γ,
- Φ( ), Ψ(Δ,), Ψ(, Γ),
- Ψ(,),

as well as Φ(Δ), Ψ(Δ,Γ) (no holes, like in the case of Δ and Γ).

These things are *not* in the (Holy) Script Itself, they are only used *to speak about* the Script. That's why we call the corresponding letters "metavariables", nowadays. To Frege they are *Buchstaben*, pure and simple.

[23]To be accurate, the Comma of (to be used in) the (Fregean) Script [BS] should be different from the comma used to speak about the Script itself. For obvious reasons, the first one – a simple meta-comma –, cannot be a (Classical, Old) Greek comma [?], it can be only Latin (or German, at most), while the second one – a meta-meta-comma – should be a honest English comma — unless I'd revert to German for the rest of this paper, like at the very beginning, in the title (this would look, however, strange, and rather impolite to most of my readers, I suspect). Should I use, perhaps, invisible [meta-meta-] quasi-quotes for the last one, and claim explicitely it's in (Middle) English? Hard to say... Noting – for the record – the fact that this points out to a serious oversight in the current practice of (both German and American) mathematics and (mathematical) logic – as well as in the design of TEX, and AMSTEX, for that matter –, I will let my friends, the True meta-Quineans of Boston – and Cambridge – MA, the Distinguished Officers of the AMS Governance – http://www.ams.org/about-us/governance/governance, etc. –, the Council Members, Committees, and Representatives of the ASL – http://www.aslonline.org/info-governance.html, http://www.aslonline.org/info-council.html –, Professor Donald E. Knuth – http://www-cs-faculty.stanford.edu/ uno/vita.html –, and, possibly, the *Bundestag* – http://www.bundestag.de/ –, to decide about, in this intricate affair, promising to introduce all due corrections in the next version of this paper, once they have reached, democratically, *the final solution* (according to the Constitution of the Association for Symbolic Logic [ASL] – http://www.aslonline.org/info-about-constitution.html –, the fundamental text: Willard van Orman Quine, **Mathematical Logic**, Harvard University Press, Cambridge, MA, 1940 (revised edition ed. 2003 [ISBN 0-674-55451-5]), and, possibly, the *Grundgesetz für die Bundesrepublik Deutschland* [sic] – http://www.gesetze-im-internet.de/bundesrecht/gg/gesamt.pdf, etc.). As for the common (round) parentheses, I will pretend they're international (see, e.g., the current usage in the Charter of the United Nations – http://www.un.org/en/documents/charter/ [in English, with links to the MSA, Chinese, French, Russian, and the Spanish versions, on the same page] –, if in doubt), so they won't ultimately deserve quoting – nor translation –, anyway.

In modern parlance, we use to say that

- Δ, Γ *range over* Proper Names,
- Φ *ranges over* Concept-Names, and
- Ψ *ranges over* [Two-Place] Function-Names,

but, although he has something as *the range of a letter* (to us: metavariable) in BS and GGA, Frege would not allow us to write down things like Φ and Ψ, without Holes – markers for non-saturation or incompleteness – and recommends filling in the Holes of $\Phi(\ )$ and $\Psi(\ ,\ )$ with the Grk-LC-C-letters reserved for Place-Holders, ξ and ζ resp., like in: $\Phi(\xi)$ and in $\Psi(\xi,\zeta)$, resp.

On the other hand, since I introduced already a (meta-meta-) notation (sic) for sub-kinds, there is no need to copy the original Greek (meta-) notation from BG and GGA, anymore; we can just "declare metavariables" (and their "syntactic sub-kinds") of our own, in the usual way.

Moreover, I shall conveniently adopt a so-called "autonymous usage" for metavariables (confusing thus, deliberately, Boston and 'Boston' – or ⌜Boston⌝, perhaps –, at meta-level). This is just to spare on (meta-) quotes, of course.

Metavariable declarations. We use the convenient (meta-meta-meta-) notation "... :: ..." in order to put things on (meta-) paper.

Let x, y :: Grk-LC-C, a, b :: Grk-UC-C-O, and F, G :: Grk-UC-C-F.

We can have F(x) and G(x,y), thus, in place of things with Fathers, Sons, and Holes, and spare on Greek letters, with the same occasion[24].

Samples of autonymous (meta-meta-) usage:

- the letter F stands for an arbitrary Concept (a Function with a single Argument-Place) indicated by the letter x, as in F(x);
- the letter G stands for an arbitrary Function with two Argument-Places, indicated by x and y, resp., in this order, as in G(x,y).

With an additional specification on Locatives, says Frege: we are only allowed to write F(x), not a mere F, and similarly, for G's: we are only allowed to write G(x,y), not G or G(x).

Yet, there is no need to (meta-meta-) quote the (meta-) letters F, G, x, and y, this time.

Proto-Substitution. From the previous explanations, we know already that we are allowed to insert a for x, and a,b for x,y resp. in the corresponding Argument-Places / Positions (that is: fill in the holes).

[24]So that we can use them, for other purposes perhaps, later on.

In such cases, we can also record on (meta-) paper what we have done, by using yet another piece of (meta-meta-) notation, as a shorthand for would instructions:

- "put the Object-Letter a in place of the Place-Holder x in a Concept-Name F(x)", and
- "put the Object-Letters a, b – in this order – in place of the Place-Holders x, y in a Function-Name G(x,y)", resp.,

whereupon we can possibly think of them as instructions to perform operations on Letters (*Buchstaben*, to us: metavariables).

Such (meta-meta-) notations together with the corresponding results can be then written as *inductive definitions*:

- F(x)[x:=a] $=_{def}$ F(a), and
- G(x,y)[x:=a,y:=b] $=_{def}$ G(a,b), resp.,

where the outcomes F(a), G(a,b) can be called "result of substituting a for x in F(x)", and "result of substituting a for x and b for y in F(x,y)", resp., taking also ("inductively") into account the fact that we have specified already the corresponding *ranges* of all the Letters therein involved.

'Sophroniskos is the Father of Socrates' should illustrate both cases.

This is a modern, rather recent, habit, however. The corresponding would-be operation is usually called "substitution", resp. "simultaneous substitution", and the suffix "...[x:=a]" reads colloquially "x becomes a in...", etc.

Frege doesn't bother to invent such (meta-meta-) notational subtleties: too obvious, of course[25]. The basic idea (*Proto-Substitution*, or *Substitution for Object-Arguments*) is *controlled semantically*, so to speak, from his point of view, and follows from the explanations, anyway.

But wait! What kind of thing is the outcome – F(a), and G(a,b) –, after all?

Obviously, 'Sophroniskos is the Father of Socrates' is something written down on paper (it is an expression called "proposition", usually, and a Name, in Frege's terms), because it appears between (Frege) quotes. Yet, according to the Proto-Wisdom (*Ur-weisheit*) referred to earlier, there are only *two* Kinds of Names on

[25] He would have, certainly, poked a lot of fun on our curious notational [meta-meta-] habits.

Earth[26]: Names of Objects and Names of Functions[27]. And the Name above does not seem to fit either of those Kinds.

Frege's answer to the question is at least surprising, at a first look: the Name 'Sophroniskos is the Father of Socrates' is a Name for *The Truth* (*das Wahre*).

For the Truth is an Object, and so should be The Falsehood (*das Falsche*). Both $\top$ and $\bot$ are Objects "out there", for Frege, remember?

Whence a True Proposition is a Name for The Truth, the Object $\top$, and, symmetrically, a False Proposition is a Name for *das Falsche*, the Object $\bot$.

Propositions are Proper Names, thus, like 'Sophroniskos', 'Socrates', 'Abū ᶜAlī al-Ḥussayn ibn ᶜAbd-Allāh ibn Sīnā' ('Avicenna', for short), 'William Shakespeare', 'Gottlob Frege', 'Rose', 'Alice', 'Diana', 'Venus', and, even '40 Eridani A', as well as '0', and '1', actually, except for the fact that Propositions are denoting The Truth ($\top$) or The Falsehood ($\bot$), instead of people, cats, planets, stars, or numbers.

This way of manufacturing Propositions from Proper Names and Function-Names works with 0 and 1, as well, in place of Sophroniskos and Socrates (or Mr White Sr. and Mrs Rose Black – the sister of Miss Alice White –, for a change): for instance '$0 = 0$' is going to be a Name of The Truth ($\top$), and so is '$1 = 1$', no need for Fathers, Sons or Daughters[28]. And, on the same line of thought, '$0 = 1$' should be a Name of The Falsehood ($\bot$), of course.

Note the indefinite article ("a") before "Name", in the above[29]: like my *Blue Rose*[30], in fact,

Frege's Truth (*das Wahre*, *The Truth*, or $\top$) has many, many (Proper) Names.

[26]*Pace* his frequent references to (The Planet) Venus, Frege doesn't really care about Naming Conventions on other planets in our solar system, nor about those that might be in force on would-be other planets, located in systems like Canopus, Sirius, and 40 Eridani A, B, C, for that matter. (This is, likely, one of the reasons why Mr Isaac Asimov, Mrs Doris Lessing, and Mr Gene Roddenberry never referred to him, alas.)

[27]The Kinds (*Arten*) Themselves need *not* be on Earth. Nor on (Meta-) Paper, for that matter. At least for a while.

[28]Nor for Lady Di.

[29]The name of a Name is a not a mere name, of course. For instance, a (Proper) Name of 'Name' is ' 'Name' ' – possibly ⌜⌜Name⌝⌝ –, not 'Name' itself.

[30]Miss Alice White, *scilicet*.

2 The Logic of *Grundgesetze*

Die Frage nun, warum und mit welchem Rechte
wir ein logisches Gesetz als wahr anerkennen,
kann die Logik nur dadurch beantworten,
dass sie auf andere logische Gesetze zurückführt.
GOTTLOB FREGE **Grundgesezte i, 1893,** *Vorwort*, p. xvii

The New Rules of Logic

Unlike the Script of his booklet[31] of 1879 [BG], the Conceptual Script of the **Grundgesetze** [GGA] is supposed to record the inferential structure of *arithmetic*.

Although arithmetic *is* logic, for Frege, the primitive setting of BG (axiomatics with *modus ponens*, substitution, etc. as rules of inference) is not best suited to this purpose.

Instead of *deriving* the would-be rules of inference required to this purpose from the BG-system, Frege starts afresh with a new set of inferential rule-schemes.

On this reason, the GGA-*Logik* appears to us as a *logic of rules* (*Regellogik*), not as a mere *proposition*[*al*] *logic* (*Satzlogik*).

If we take the trouble to manufacture the appropriate notation, the Fregean design of 1893 becomes strangely familiar to the modern reader[32].

In order to make things more transparent, we consider first the quantifier-free fragment of the GGA-logic[33].

To begin with, we fix the terminology and some notation, keeping always an open eye on would-be equivalences subsisting between the current way of speaking in logic and Frege's own terms.

Propositions containing implication (Frege's *Bedingungsstrich*) will be printed horizontally (A $\rightarrow$ B).

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[31] Frege's word (*Büchlein*).

[32] The parallel with the so-called "natural deduction" systems and / or the sequent-presentations of logic (Paul Hertz 1921–1929, Stanisław Jaśkowski 1926–1934, Gerhard Gentzen 1933–1934) has been already noticed by several authors in print (as, e.g., Tichý 1988, von Kutchera 1996, Schröder-Heister 1997, 1999, etc.). In retrospect, it is amazing that neither Jaśkowski, nor Hertz – not even Gentzen – would refer to GGA. Jaśkowski 1934 quoted only BG, and obliquely so, just in order to show that two of the axiom-schemes of BG – i.e., (K) and (S) below – are equivalent to one of his "supposition rules" (conditionalization / the deduction theorem), if *modus ponens* is present. His "supposition calculus" – a system of "natural deduction" (Gentzen's term) – is, in fact, different from Frege's GGA-logic, in that he makes from the very beginning distinctions Frege won't have recognized as legitimate. On the other hand, although Gentzen 1934 did not mention Frege either, his sequent presentation of logic is based, obviously, on GGA (by pondering also on Hertz 1922, 1923, 1928, 1929, 1929a, perhaps).

[33] Technically, both the BG- and the GGA-logic include the so-called "quantifiers". The word – although not the concept – comes from Charles S. Peirce, cca 1880.

A Fregean *conditional* is a proposition (*Satz*) of the form

$$A_1 \to .A_2 \to ... \to .A_n \to C, [n \geq 1],$$

with a *succedent* (called *Oberglied*) C, and one or more *antecedents* (its *Unterglieder*) A_1, ..., A_n[34]. Obviously, in the propositional fragment, the *Oberglied* can be only an *atom* (a *propositional variable*), in our terms, or a negation (printed ¬A, here).

We collect conveniently the *Unterglieder* in a sequence $\Gamma := (A_1, ..., A_n)$, so that the above becomes

$$\Gamma \to C \equiv (A_1, ..., A_n \to C), [n \geq 1]$$

(where ≡ stands for syntactic identity). On practical reasons, we might also need an additional convention to the effect that $\Gamma \to C \equiv C$, for n = 0, in the above.

For asserted conditionals, $\vdash \Gamma \to C$ is *the same thing as* $\vdash (A_1 \to ... \to .A_n \to C)$, so the conventions above are meant to save printing →'s, and paretheses (and / or separating dots), mainly.

A Fregean *transition* (*Übergang*) is a meta-conditional of the form

$$\vdash \Gamma_1 \to C_1, ..., \vdash \Gamma_k \to C_k \Rightarrow \vdash \Gamma \to C$$

with, usually, k := 1, 2, corresponding to what we call *rules of inference*.

In order to state the Fregean transitions in pure "syntactic" terms, we shall introduce a few systematic abbreviations.

Notation (*Shuffling*). Where Γ is a finite sequence (here: a sequence of propositions) and i is a non-negative integer, $\Gamma *_i A$ reads "insert A at place i in Γ".

This operation can be defined by an obvious induction[35], as follows:

(0,i) If Γ is empty (length 0), then $\Gamma *_i A \equiv A$, for any i, else,
(n,i) let $\Gamma := (A_1, ..., A_n)$, $n \geq 1$ (Γ has length n); then

(n,0) $\Gamma *_0 A \equiv (A, A_1, ..., A_n)$,
$(n,i_{<n})$, $\Gamma *_i A \equiv (A_1, ..., A_i, A, A_{i+1} ..., A_n)$, for $\leq 1 \leq i \leq n-1$, and
$(n,i_{\geq n})$, $\Gamma *_i A \equiv (A_1, ..., A_n, A)$, for $i \geq n$.

In other words, location 0 is "in front of" the elements of Γ, while location $i \geq 1$ is "after" the i-th element of Γ, up to $i \leq n-1$, and "after" its last element, if $i \geq n$.

[34] In a horizontal arrangement, the corresponding terms would have been, likely, *Vorder-* and *Hinterglied*, resp.

[35] Specifically, by induction on pairs (m,i), for any two non-negative integers m, i.

One can also extend this notation to pairs of finite sequences $\Gamma_1 *_\sigma \Gamma_2$, for an appropriate (finite) sequence of non-negative integers, $\sigma := (i_1, ... i_m)$, where m is the length of Γ_2[36].

We shall apply the notation above to asserted Freagean propositions,

$$\vdash \Gamma \to C\ (\equiv A_1 \to ... \to .A_n \to C),\ [n \geq 1],$$

where C is the succedent (*Oberglied*), and the A_i's (elements of Γ) are the antecedents (*Unterglieder*) of the proposition $\Gamma \to C$, by writing, e.g.,

$$\vdash \Gamma *_i A \to C,$$

for the asserted proposition obtained from $\Gamma \to C$, by inserting A at "place" i in Γ:

$$\vdash \Gamma *_i A \to C\ (\equiv A_1 \to .A_2 \ ... \to .A \to ... \to .A_n \to C),$$

and analogously for $\vdash \Gamma_1 *_\sigma \Gamma_2 \to C$, for an appropriate shuffle-subscript σ.

Conventions. If Γ is as above, then Γ_Π stands for an abitrary permutation of Γ, while Γ_W stands for the sequence obtained from Γ by deleting all duplicates of its elements, if any (so that only first occurrences are retained).

These conventions are supposed to apply, *mutatis mutandis*, to asserted Fregean propositions $\vdash \Gamma \to C$, as well.

Notation. Finally, in order to save some (meta-) talk, while re-stating the Fregean transitions called *Wendung*(*en*), – our usual *contraposition* rules – we set $A^\perp := \neg C$, if $A \equiv C$, and $A^\perp := C$, if $A \equiv \neg C$, where $\neg C$ stands for (the primitive Fregean) negation[37].

We can now turn to the proper logical ingredients of GGA[38].

There is, first, an axiom (scheme), and a neutral group of transitions (rule-schemes in fact, analogues of what most people would call, *mutatis mutandis*, "structural rules", nowadays):

(id) $\vdash \Gamma *_i A \to A$,

[36] Intuition: although not recommended in the game of poker and the like, card-shuffling can be performed sequentially, by inserting one card at a time, *at an arbitrary place*, in a half-deck of cards. In particular, supressing the i- and σ-subscripts on the shuffle-operation * amounts, ultimately, to the fact that sequences so described are to be taken modulo arbitrary permutations (i.e., if "context-free", they would become so-called *multisets*).

[37] This is mainly because the Fregean *Wendung* is stated as a single rule (-scheme) in terms of truth-values in GGA:I, §48, p. 61. The notational convention above yields the closest "syntactic" schematic analogue of Frege's (meta-) statement *ad locum*: a single (meta-) statement in place of four; see below.

[38] On obvious reasons, the order of exposition following below is not that of GGA.

$(\text{prm})\ \vdash \Gamma *_i \text{A} \to \text{C} \Rightarrow\ \vdash \Gamma *_j \text{A} \to \text{C},$
$(\text{ctc})\ \vdash \Gamma *_i \text{A} *_j \text{A} \to \text{C} \Rightarrow\ \vdash \Gamma *_k \text{A} \to \text{C},$

for any sequence $\Gamma := (A_1, ..., A_n)$, $n \geq 0$, and all i, j, $k \geq 0$.

As regards the terminology, the rule-scheme (prm) – said, more or less, "permutation" or "exchange" in modern terms – is called *Vertauschbarkeit der Unterglieder* in Frege[39], while the (ctc)-scheme – our "contraction", more or less – is called "fusion" (*Verschmelzung gleicher Unterglieder*)[40].

In matters of propositional axioms, Frege states actually[41] only two instances of (id), viz. those obtained by instantiating (1) $\Gamma := (A, B)$, with $i = 0$, and (2) $\Gamma := (A)$, with $i = 0$, resp., idest

(K) $\vdash A \to (B \to A)$, and
(I) $\vdash A \to A$,

as would-be *laws* (*Grundgesetze*), in contrast with the mere *Regeln*, stated and discussed at length separately, but we can easily obtain the rest, including the more general *dilution* principle (called also "weakening", or "thinning", or even "irrelevance", by recent writers on allied topics),

$(\text{dil})\ \vdash \Gamma \to \text{C} \Rightarrow\ \vdash \Gamma *_i \text{A} \to \text{C},$

for any sequence $\Gamma := (A_1, ..., A_n)$, $n \geq 0$, and all $i \geq 0$[42].

From the above, one can easily obtain, with our global (meta-) notation for permutations and fusions,

$(\text{prm*})\ \vdash \Gamma \to \text{C} \Rightarrow\ \vdash \Gamma_\Pi \to \text{C},$
$(\text{ctc*})\ \vdash \Gamma \to \text{C} \Rightarrow\ \vdash \Gamma_W \to \text{C},$

for any sequence $\Gamma := (A_1, ..., A_n)$, $n \geq 2$.

In general, Frege's (propositional) *Gesetze*, as well as his transitions are to be taken modulo (prm*) [exchanges/permutations] and (ctc*) [fusions/contractions].

Worth noting separately are the following instances of (prm*) and (ctc*), "limit cases", so to speak:

[39] GGA:I, §14, 26, and §48, p. 61, Rule (2).

[40] GGA:I, §15, p. 29, and §48, p. 61, Rule (4).

[41] GGA:I, §18, p. 34, §47, p. 61.

[42] Obtain first the single-premiss rule-form [K], of (K), with *modus ponens* – an instance of (cut) below –, substitute, and apply then (prm) in order to get (dil). The fact that Frege meant something like (id), in general, instead of (K), is obvious from GGA:I, §18, p. 34, for instance, where (I) is said to be *ein besonderes Fall* of (K).

$[C]\ \vdash A \to .B \to C \Rightarrow \vdash B \to .A \to C,$
$[W]\ \vdash A \to .A \to C \Rightarrow \vdash A \to C,$

where [C] is the one-premiss rule-form of the (otherwise redundant) axiom-scheme (C) of BG[43], and [W] the one-premiss rule-form of (W), also obtained, at least implicitly, in BG[44]:

$(C)\ \vdash (A \to .B \to C) \to (B \to .A \to C),$
$(W)\ \vdash (A \to .A \to C) \to (A \to C).$

Next, there are two more transitions called "inferences" (*Schlüsse*, *Schlussweisen*):

$(\text{cut})\ \vdash \Gamma *_i A \to C, \vdash \Gamma' \to A \Rightarrow \vdash \Gamma *_\sigma \Gamma' \to C,$
$(\text{syl})\ \vdash \Gamma *_i B \to C, \vdash \Gamma' *_j A \to B \Rightarrow \vdash \Gamma *_\sigma \Gamma' *_k A \to C,$

for any two sequences $\Gamma := (A_1, ..., A_n)$, $\Gamma' := (B_1, ..., B_m)$, with m, n ≥ 0, i, j ≥ 0, and an arbitrary shuffle-index σ.

The first transition[45] corresponds, more or less, to our "cut" (*Schnitt*, following Gentzen). The special case of (cut) with m = n = 0 is just *modus ponens*, of course. Repeated applications of (cut) yield an obvious "multicut", a rule-scheme with k+1 premisses, for k ≥ 2.

The second transition (syl) corresponds to the traditional syllogism, and is stated explicitly modulo (ctc*) in GGA[46]. One can generalize (syl) to a would-be "poly-syllogism", corresponding to the traditional *Kettenschluss*, a rule-scheme with k+1 premisses, for k ≥ 2, as for (cut), but Frege doesn't seem to appreciate such complex transitions: his rules of inference have at most two premisses[47].

All this is redundant, of course, since, one can easily have (syl) from the general form of (cut) above. However, Frege seems to prefer a primitive (cut)-scheme with empty Γ', so that (syl) must be stated separately.

If we intend a more economical primitive setting, one can choose only (id), (dil) and a special case of (cut), with Γ = Γ', incorporating already the effect of (ctc):

$(\text{cut}_W)\ \vdash \Gamma *_i A \to C, \vdash \Gamma \to A \Rightarrow \vdash \Gamma \to C,$

[43] BG, §16, proposition (8).

[44] In BG, §16, proposition (11). Actually, (11) is $\vdash (A \to .A' \to C) \to (A' \to C)$.

[45] Cf. *6. Schliessen (a)*, in GGA:I, §48, p. 62, etc.

[46] Cf. *7. Schliessen (b)*, in GGA:I, §48, p. 62.

[47] The general "syllogism" appears first in Hertz 1922–1923 – where it is wieved as a generalization of *modus* BARBARA – together with (dil) and (id). Incidentally, Hertz never refers to Frege's logic and to Frege in general (except once, to the **Grundlagen der Arithmetik**, but this *not* in a logic paper, and the reference has no bearing to logic, anyway).

wherefrom one can obtain easily (ctc), as well as (syl)[48]. But economy is not a concern in GGA, anyway.

In view of would-be further simplifications, one might note the following special instances of (syl):

[CoBB] $\vdash$ A $\to$ .B $\to$ C, $\vdash$ A' $\to$ B $\Rightarrow$ $\vdash$ A $\to$ .A' $\to$ C,
[S] $\vdash$ A $\to$ .B $\to$ C, $\vdash$ A $\to$ B $\Rightarrow$ $\vdash$ A $\to$ C,
[B] $\vdash$ B $\to$ C, $\vdash$ A $\to$ B $\Rightarrow$ $\vdash$ A $\to$ C,
[CB] $\vdash$ A $\to$ B, $\vdash$ B $\to$ C $\Rightarrow$ $\vdash$ A $\to$ C,

and the fact that [S] follows from [CoBB] and [W] above.

Notation (*Witness grammar*). Instead of the global Fregean separators (*Abzeichen*) marking applications of (cut) and (syl), we shall use next a more specific witness notation, described as follows:

(0) if X is a witness for (A $\to$ C), and Y is a witness for A, we write X $\triangleright$ Y, for the result of applying *modus ponens* to X and Y, i.e. for C;

(1) if X is a witness for (B $\to$ C), and Y is a witness for (A $\to$ B), we write X $\circ$ Y, for the result of applying rule [B] to X and Y, i.e. for (A $\to$ C),

and, analogously,

(2) if X is a witness for (A $\to$ .B $\to$ C), and Y is a witness for (A $\to$ B), we write X $\square$ Y, for the result of applying rule [S] to X and Y, i.e. for (A $\to$ C).

This notation is to be taken taken modulo uniform substitutions[49].

The "witness operations" corresponding to *modus ponens*, [B], and [S] can be thus viewed as binary operations on appropriate witness labels X, Y.

In particular, where (S) is a witness for proposition (2) of BG, and $\triangleright$ is as above,

(S) $\vdash$ (A $\to$ .B $\to$ C) $\to$ (A $\to$ B $\to$ .A $\to$ C),

[48] Gentzen 1933 noticed that Hertz's general form of (syl) can be obtained from (id), (dil), and (cut_W). His main concern was different from that of Hertz, however, so he preferred, to have, *mutatis mutandis*, a primitive setting with (I), and the "structural rules" [Gentzen's term] (dil), (prm), (ctc), and (cut). Like Jaśkowski, nearly a decade before (most of Jaśkowski 1934 is based on results obtained in 1926), Gentzen 1934 then distinguished between deductions or inferences and conditional propositions, written as implications (so that he could also state conditionalization / the *deduction theorem*), supplied the [now] missing "logical rules" and then went to show that (cut) is redundant, etc. This would have likely seemed strange to Frege!

[49] The corresponding "operations" on witnesses represent thus rules of inference (modulo substitution), and are order-sensitive, of course. The above amounts to an applied "combinatory logic" notation for proofs / derivations. We don't need the "characteristic equations" of the witnesses (I), (K), (S), (B), (CB), however; we have just a "witness grammar", a convenient notational tool, *à la* Carew A. Meredith, say. Cf. e.g., Rezuş 1982.

$[\triangleright]\ \ \vdash A \to C, \vdash A \Rightarrow \vdash C,$

the two-premiss rule [S] amounts to $[S](X,Y) := X \,\Box\, Y = (S) \triangleright X \triangleright Y$, of course.

Likewise, writing [K] – resp. [K](X) – for the single-premiss rule obtained with *modus ponens* from (K),

$[K]\ \ \vdash A \Rightarrow \vdash B \to A,$

i.e., $[K](X) := (K) \triangleright X$, one has *by definition*, for appropriate X and Y, $[B](X,Y) := X \circ Y = [K](X) \,\Box\, Y$, as well as $[CB](X,Y) := Y \circ X$, and $[B](X,Y) := [CB](Y,X)$, modulo (prm), etc.

Examples.

(1) Given the rule [S] and the axiom-scheme (K) we can have (I), that is $\vdash A \to A$, above, by applying [S] to appropriate substitution instances in (K), i.e., $(I) = (K) \,\Box\, (K)$.

(2) Analogously, sparing on square brackets and parentheses (by associating to the left, say) – with K(X) for $[K](X) = (K) \triangleright X$, for instance, as above –, the rather lengthy Łukasiewicz 1934 derivation of the BG axiom (C) [= BG, proposition (8)] from (K) and (S), with *modus ponens* and substitution, simplifies to $(C) := K(CBK) \,\Box\, (S)$, where $(CBK) := K(S) \,\Box\, K \,\Box\, K(K)$, condensing thus half a page (in print) to a single line[50].

So, in general, deductions become *explicit definitions*, actually.

Remark (*Conditionalization*). The careful reader has already noticed, by now, the fact that, with this minimal equipment – idest granted (K) and rule [S] –, one can already obtain the usual *conditionalization* rule, or the so-called *deduction theorem*[51]. On doctrinal reasons, this rule cannot be stated in the Script, however. For Frege a statement $\vdash (A \to B)$ is, in fact, *the same thing* as *the deduction of* B *from the assumption* A, and a *law* (*Gesetz*) *of logic* is just a *codification of a rule of inference* (or else *a codification of a package of such rules*).

[50] The original derivation of Łukasiewicz 1934 – a typical piece of (proto-) "Polish logic" – amounts, actually, to $(C) := (S) \triangleright K(S) \triangleright (K) \triangleright (CBK) \triangleright (S)$, i.e., it is not "(S)-(K)-normal", so to speak. As a matter of historical detail, Carew A. Meredith, the main promotor of the ("condensed") proof-style exemplified here, has learnt this kind of "witnessing" proofs / derivations by *modus ponens* (and substitution) from his mentor, Jan Łukasiewicz, while attending the Pole's lectures in Dublin, some time after the Second World War, around the early fifties. Cf., e.g., Meredith 1977, for bio-biographical details on C. A. Meredith (stemming from one of his American cousins who happened to be a logician, too), and Rezuş 1982, Kalman 1983 (and the abstract Kalman 1974), Hindley & Meredith 1990, Hindley 1997, etc., for further information on the so-called "condensed detachment" operator of C. A. Meredith. For a recent discussion of closely related topics, going, otherwise, beyond the scope of the present notes, see, e.g., the Copenhagen Lectures of Sørensen & Urzyczyn 1998, and 2006 [minutely revised].

[51] In combinatory logic, the λ-abstractor can be defined in terms of (S) and (K). In fact, one needs only (I) and two operations: a first one corresponding to rule [K], the one-premiss rule from (K), and a second one corresponding to [S].

Finally, there are two other transitions, serving to manipulate negation, the first one, called *Wendung*[52], corresponds to *contraposition*, and the last one to a form of classical *dilemmatic reasoning*[53]:

(ctp) $\vdash \Gamma *_i \mathrm{A} \to \mathrm{C} \Rightarrow \vdash \Gamma *_j \mathrm{C}^{\perp} \to \mathrm{A}^{\perp}$,
(abs) $\vdash \Gamma *_i \neg\mathrm{A} \to \mathrm{C}, \vdash \Gamma *_j \mathrm{A} \to \mathrm{C} \Rightarrow \vdash \Gamma \to \mathrm{C}$.

Of course, (ctp) amounts to four distinct *contrapositions*, in our terms[54]:

(ctp B) $\vdash \Gamma *_i \mathrm{A} \to \mathrm{C} \Rightarrow \vdash \Gamma *_j \neg\mathrm{C} \to \neg\mathrm{A}$,
(ctp $\hat{B}$) $\vdash \Gamma *_i \neg\mathrm{A} \to \neg\mathrm{C} \Rightarrow \vdash \Gamma *_j \mathrm{C} \to \mathrm{A}$,
(ctp C)) $\vdash \Gamma *_i \mathrm{A} \to \neg\mathrm{C} \Rightarrow \vdash \Gamma *_j \mathrm{C} \to \neg\mathrm{A}$,
(ctp $\hat{C}$)) $\vdash \Gamma *_i \neg\mathrm{A} \to \mathrm{C} \Rightarrow \vdash \Gamma *_j \neg\mathrm{C} \to \mathrm{A}$.

Our (meta-) notation with $\ldots^{\perp}$'s yields a "syntactic" transcription of Frege's semantic formulation of the rule-scheme called *Wendung*, in terms of truth-values (*Wahrheitswerthe*).

The rule-scheme (abs) is stated, again, modulo (ctc), in GGA.

It yields the cognate rule *consequentia mirabilis* of Gerolamo Cardano (1570), also known as *the Rule of Clavius*[55]:

(clv) $\vdash \Gamma *_j \neg\mathrm{A} \to \mathrm{A}, \Rightarrow \vdash \Gamma \to \mathrm{A}$.

In fact, given (id), on the one hand, and (ctp $\hat{C}$), (syl), on the other, (abs) and (clv) turn out to be equivalent.

Whence, ultimately, the following list contains a very compact, non-redundant, and complete set of rules for classical (propositional) logic:

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[52]Cf. *3. Wendung*, in GGA:I, §48, p. 61, cf. also §14, p. 27, etc.

[53]Cf. *8. Schliessen (c)*, in GGA:I, §48, p. 62, etc.

[54]Cf. GGA:I, §14, p. 27, etc.

[55]Apparently, Gerolamo Cardano (1501–1576) got it on his own, and he was rather proud of this (see, e.g., his treatise **De proportionibus**, Basle 1570, in Cardano 1663, **4**, where this proof-pattern is qualified, p. 579, *res admirabilior quae inventa sit ab urbe condito. . ., longe majus Chrysippaeo syllogismo. . .*, etc.). The Jesuit Kristoph Klau – Christophorus Clavius SJ (1537–1612) or *Pater Clavius*, in Latin – noticed, however, a prior usage of the *mirabilis argumentandi modus* in Euclid's **Elementa** [*etiam usus est Euclides*, a comment *ad* Eucl., IX.12], and – by confusing use and mention (!), as well as some other things – dismissed the priority claim of the Pavian (cf. Clavius **Opera mathematica I**, Mayence 1611, *ad loc.*, in **1.2**, page 11; and, possibly, his comments *ad* Theodosius, in ibid., **I.12**). Subsequent learned Jesuit propaganda induced scholars to credit Clavius with the corresponding "Law", later on. As a matter of fact, *Pater Clavius* missed other "antecedents" in this respect, as well, e.g., Aristotle, in his **Protrepticus**, now lost, and the Stoics, as reported by Sextus Empiricus **Adv. math.**, VIII, 281–2, 466 *et sq.* (Otherwise, by such scientific standards, one should also credit our ancestor Lucy with the "discovery" of *modus ponens*.) For further details on *consequentia mirabilis* and the like see Kneale 1957, Kneale & Kneale 1971, Miralbell 1987, Nuchelmans 1991, 1992, Bellissima & Pagli 1996, etc.

- a single axiom-scheme: (K),
- two (cut) rules: *modus ponens*, and
- rule [S] (alternatively, [S] can be replaced by [CoBB] and [W]),
- two contrapositions: (ctp C), and (ctp $\hat{C}$),
- rule (abs) (or, alternatively (clv), as noted above),

and it is easy to see that one cannot make any further reductions, i.e., one has already a non-redundant rule-system, indeed.

Otherwise, there are many possible variations on this theme. Yet, as already mentioned above, the economy of means is not an issue in GGA.

Consistency has been already established in GGA (where the rules are first justifed "semantically", so to speak).

For completeness it is enough to derive the postulates (axioms and rules) of a system for which we do already have the result (!). In order to obtain the axiom system of BG, for instance, one should remember the Remark on conditionalization, above, and notice the fact that the double negation rules:

$[\Delta^{\perp}]$ $\vdash \neg\neg\mathrm{A} \Rightarrow \vdash \mathrm{A}$,
$[\nabla^{\perp}]$ $\vdash \mathrm{A} \Rightarrow \vdash \neg\neg\mathrm{A}$,

can be obtained from (I) and (ctp C), (ctp $\hat{C}$), whence also (ctp B), and (ctp $\hat{B}$).

Finally, closing under conditionalization yields:

(K) = BG (1) $\vdash \mathrm{A} \rightarrow .\mathrm{B} \rightarrow \mathrm{A}$,
(S) = BG (2) $\vdash (\mathrm{A} \rightarrow .\mathrm{B} \rightarrow \mathrm{C}) \rightarrow (\mathrm{A} \rightarrow \mathrm{B} \rightarrow .\mathrm{A} \rightarrow \mathrm{C})$,
($B^{\perp}$) = BG (28) $\vdash \mathrm{A} \rightarrow \mathrm{C} \rightarrow .\neg\mathrm{C} \rightarrow \neg\mathrm{A}$.
($\Delta^{\perp}$) = BG (31) $\vdash \neg\neg\mathrm{A} \rightarrow \mathrm{A}$,
($\nabla^{\perp}$) = BG (41) $\vdash \mathrm{A} \rightarrow \neg\neg\mathrm{A}$,
(modus ponens) $\vdash \mathrm{A} \rightarrow \mathrm{C}, \vdash \mathrm{A} \Rightarrow \vdash \mathrm{C}$.

ı.e., the full propositional logic of BG. As completeness (for BG) has been aleady established by Łukasiewicz 1931, we have completeness for the GGA *Regellogik*, as well[56].

[56] Here, as observed by Jan Łukasiewicz 1934, (C) [= BG, proposition (8)] is redundant. A somewhat simpler derivation has been already mentioned above, as an example of "witnessing". Moreover, the resulting axiom system – without (C) – is independent, as shown, for instance, by Christian Thiel, in his Erlangen Dissertation (1965). Cf. the translation Thiel 1968, p. 21.

3 Appendix. *The Name of a Rose*

Damit ein solches Unternehmen Erfolg haben könne,
müssen natürlich die Begriffe, deren man bedarf, scharf gefasst werden.
GOTTLOB FREGE **Grundgesezte i, 1893,** *Einleitung,* p. 1

...a rose by any other name would smell as sweet...

So Juliet, "Daughter to" Capulet, and William Shakespeare: a Rose is a rose – *as sweet as any other Rose* –, even if (she is) called by any other name.

And what if she is a young lady called Rose? "The name of a Rose, if called by her own name, is Rose!' – my logic textbook says.

What if she's called by any other name? "The name of Rose, if called by any other name than her own (name), is still Rose!' – insists the book.

What about *Roses Called by Something Else Than Names*? What about a *Blue Rose*, for instance? My neighbour, if she's *Dressed in Blue*, a Rose by her own (name). Or else *Blue Rose*, for instance, a fancy white rose to me, she is (a) White, indeed, although no Rose, in fact, as she's called so *by her blue eyes* alone. So she is a *White Rose*, too, this even by her (own) name, the fancy rose, *Blue Eyes* and so.

*

In front of The Mirror, near Diana – that's Lady Di, The Cat, my cat –, there is a bunch of sweet RED Roses.

Just got them fresh, this morning, from Alice Blue Eyes, the sister of my neighbour, Rose.

*

Miss White and Mrs Black are twins. They like colors and roses most of all (the father is a painter, the mother a fashion-designer — this is not a reason, though).

They used to be both White, exactly like their parents – like any (other pair of) twins 'round the world, in fact – but Rose married Dr. Black, the mathematician, not too long ago.

Alice got blue eyes, Rose's are black. (So, while in school, and later, among friends and so, they were able to fool people only from far.)

Since they were kids, I used to call them by names of roses, and we played the Game of Names. As Alice was and is still no rose, but only a name, I called her White Rose or Blue Rose. Her sister was, sure, a White Rose by her own name, but most of the time, Black Rose, in view of her eyes, mainly. Anyway, Rose was

Rose White, too, in all kind of scripts, including her passport (the girls were not inscribed in their parents' passports, just in case), but this was not too entertaining, for us, because she couldn't read by then.

After a while, the twins became experts in the Game of Names, and even tried to fool me by asking tricky questions, using also the talents of their mother to reinforce the effect, by dressing themselves colourfully for example, in blue or white: "What is my name, if I am dressed in blue?", asked Alice. Or else, the other one, while dressed in white, likewise. "White Rose in Blue", "Black Rose in White", I answered undisturbed.

To make things a bit more interesting (this was before the girls were able to write), I invented the Game of the Name in the Mirror. For example: Alice in the Mirror (look at her, in there!) is the name of the name of White Rose. Question: What is the name of Alice in the Mirror? Alice in the Mirror or Blue Rose?

This wasn't so simple, nor very easy anymore, mainly because each girl had a Mirror of her own, and they were able to have a look at their own names in the name of the other etc.

On obvious reasons, Mrs White, the fashion-stylist, was very angry with me at first, during a week or so – the girls were making a lot of noise around their new Mirror-Names –, but, soon after, the daughters caught her in the Mirror, too – in the new Game, I mean –, and she ended by induldging herself in endless Mirror-debates, together with the two young ladies, aged – each – six and a half. So that, after another while, her husband – my friend, the painter –, became suspicious, and inquired, cautiously nonetheless, about "that new mirror story you invented for the benefit of my family".

Once the girls have learnt to write (and count), there was no need for Mirrors anymore: they could use quotes! This wasn't easy at first, but they got the point eventually. The Game of Names reduced itself to a matter of countig quotes.

*

"*The Name of a Rose* – Alice expertly comments on Names, this morning – is *not* the name of a Rose, my friend[57]. It's the name of a book – added White Rose, Blue Eyes, in Black, for my logical comfort – by somebody else than my Rose, sweet Rose — she's *Dressed in Blue*, my Rose, sweet-rose, indeed. And *Red Rose* is *not* the name of any Rose! It is an Amsterdam café, remember? The one where we used to drink, once in a while, a coffee, and *frühstück*, last year. But don't call me, please, *White Rose*. It is *not* my name. It's even *not* the name of Rose. She's a Black now, even if a Rose, and *Dressed in White*. And I am still a White, even if no Rose, and *Dressed in Black*, this morning. *Blue Eyes*, you said? Who cares? I won't, anymore! So, don't call me *Blue Eyes, in Black*, either. It is *not* my name.

[57]Denying thereby, to my surprise – as I was unable to *hear* first her (quasi-) quotes – a Law called "A is A" or even "⌜Boston⌝ = ⌜Boston⌝" in my logic books (mainly in those printed in Boston MA).

It's not even a Name! Nor is *Black Eyes, in Blue* a name of Rose, if *Dressed in Blue*, even if she has got black eyes. *Alice-in-Black*, would do, perhaps, once in a while, for me. This morning, for instance. Like *Rose-in-Blue*, for Rose, today[58]."

*

What's the Name of (a) Rose? The Name of the One White Rose called Alice in my story, for example[59].

And, if it is *true* that the Name of a Rose is not the name of a rose – as Alice claimed –, then what should we Name this Truth?

And, before anything else: *What is a Name of Truth?* Is *The Truth* a Name of (the) Truth? Is it (the) One? – *The* Truth, perhaps? – Is it the only One?[60]

---

[58] I wonder, how should I have printed *Red Rose*, in $\mathcal{AMS}$TEX, here? With Boston quasi-quotes, perhaps? I'm not sure of this. Miss White was right! *Red Rose* is *not* a name at all, but a café.

[59] The reader should perhaps notice the fact that I didn't ask "*Where* is the Name of Rose?" (or Alice)! As a matter of fact, the latter kind of questions is uninteresting "from a logical point of view", so to speak. — On the other hand, the idea that the Name of Mrs Rose Black (or of Miss Alice White, for that matter) is, ultimately, *on paper* (possibly after printing a copy of this paper, say) is a very wrong one – a mere way of speaking, at best –, because even the Name of my Cat – Diana or Lady Di –, is, after all, an Abstract Entity, and all what we can actually put "on paper", while printing (or writing things "down"), are molecules of ink (or else graphite / "lead", allotropes of carbon). Same thing, *mutatis mutandis*, about blackboards, chalk and so (including the more recent case of so-called *e-paper* – and *e-ink* –; although technologically more sophisticated, the latter are stubbornly remaining "out there", in the so-called "physical realm"). In short, like Ideas, Concepts, and Thoughts, the Names cannot be found "out there" either. Among logicians, the Stoics and the so-called "formalists" – even some early otherwise famous Polish pioneers (like Jan Łukasiewicz, Stanisław Leśniewski, Alfred Tarski), as well as Professor Willard van Orman Quine, in more recent times – were, certainly, a bit in a hurry while locating the "syntax" next to the real Cats (and Dogs), or else at the same level as would-be traces of ink on paper, chalk on blackboards and so on...

[60] *Endnote* [June 13, 2009]. The rest of this section has been lost, mainly because of (the former) Miss Alice White. My friend is not to be blamed, nevertheless. The fact is that Miss White married Mr Green, an ET with green eyes, two years ago and, eventually, lost interest in Terran Logic, and in my lectures on Frege, (Blue) Roses and Co. She moved recently (together with Mr Green, of course) to Blue Vegas – a nice little ecological village located on a small theoretical planet in Alpha Centauri –, and took her shorthand notes (*Gabelsberger Kurzschrift*) with her. As communication with Blue Vegas is rather difficult nowadays, I was unable to ask her about the wereabouts of my Fregean ruminations, so far.

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Contents

typeset by romanianTeX © 1994–2001 Adrian Rezuş
first draft, printed in the netherlands – June 13, 2009
revised version: October 1, 2010
last revised: printed in the netherlands – May 12, 2016